Globally coupled stochastic two-state oscillators: Fluctuations due to finite numbers

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Infinite arrays of coupled two-state stochastic oscillators exhibit well-defined steady states. We study the fluctuations that occur when the number $N$ of oscillators in the array is finite. We choose a particular form of global coupling that in the infinite array leads to a pitchfork bifurcation from a monostable to a bistable steady state, the latter with two equally probable stationary states. The control parameter for this bifurcation is the coupling strength. In finite arrays these states become metastable: The fluctuations lead to distributions around the most probable states, with one maximum in the monostable regime and two maxima in the bistable regime. In the latter regime, the fluctuations lead to transitions between the two peak regions of the distribution. Also, we find that the fluctuations break the symmetry in the bimodal regime, that is, one metastable state becomes more probable than the other, increasingly so with increasing array size. To arrive at these results, we start from microscopic dynamical evolution equations from which we derive a Langevin equation that exhibits an interesting multiplicative noise structure. We also present a master equation description of the dynamics. Both of these equations lead to the same Fokker-Planck equation, the master equation via a $1/N$ expansion and the Langevin equation via standard methods of Itô calculus for multiplicative noise. From the Fokker-Planck equation we obtain an effective potential that reflects the transition from the monomodal to the bimodal distribution as a function of a control parameter. We present a variety of numerical and analytic results that illustrate the strong effects of the fluctuations. We also show that the limits $N \to \infty$ and $t \to \infty$ ($t$ is the time) do not commute. In fact, the two orders of implementation lead to drastically different results.

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I. INTRODUCTION

Stochastic on-off processes are ubiquitous in nature and include, for example, firing of neurons, spin systems where an up-spin can be labeled “on” and a down-spin “off”, and a variety of blinking phenomena such as quantum dots randomly transitioning between bright and dark (fluorescent and quiescent) states. By “stochastic” we here mean that the timing of the on-off and/or off-on transitions of the oscillators occurs at random times governed by distributions. When such oscillators are coupled together, the system may exhibit interesting global behavior. In particular, when a large number of stochastic oscillators are coupled together, the array may exhibit ordered dynamics over long or even infinite periods of time.

Discrete-state stochastic oscillator models that undergo a phase transition to synchronization or ordered behavior are frequently found in the literature and we mention just a few here. Insofar as these models are studied analytically, they all assume global coupling and consider a mean field limit. In [1] globally coupled three-state oscillators are considered as a model of stochastic excitable system. They consider a form of coupling that leads to a stable stationary state and another that shows coherent oscillations. In [2], still focusing on three-state stochastic oscillators, this work is extended to a broader class of non-Markovian systems and the resultant collective oscillations are shown to occur in large parameter regimes. Ensembles of excitable stochastic two-state oscillators with delayed feedback are studied in [3] as an abstract representation for an excitable system. Here again, depending on the properties of the single oscillators and of the global coupling of an infinite number of them, the system exhibits bistability in some cases and bulk oscillations in others. Tsimring and co-workers [4,5] have an extensive bibliography on two-state and on three-state systems of coupled stochastic oscillators that undergo a variety of interesting transitions in the mean field limit, too many to list completely here. His work has now moved on to the construction of genetic regulatory circuits based on these ideas; see e.g. [6]. Our own work on coupled three-state stochastic oscillators [7–10] has been extended in interesting ways by Assis et al. [11,12], including the discovery of a symmetry-breaking transition to a steady state that has no counterpart in equilibrium statistical mechanics.

In this paper we deal with Markovian on-off units, that is, with units for which there is a rate to transition from the off state to the on state and also a rate for the reverse. When we couple an infinite array of these units together in a particular way so as to have time-dependent transition rates that depend on the state of the array, we see a phase transition to ordered steady states as a coupling parameter is increased. Here we investigate the behavior of this system when the number of units in the array is finite rather than infinite. It is important to place our previous work on two-state oscillators in context. In [13] we began with infinite arrays of coupled two-state stochastic oscillators, coupled in the same way as we will consider here, but there we investigated the effect of changing one of the transition rates so as to include a memory, rendering
an infinite non-Markovian array. In [14] we again worked with infinite arrays of two-state units. We considered Markovian transitions from one state to the other, but the return transition was completely dictated by a refractory period. We took the transition rate and the refractory period to depend on the state of the array. The phase transition in this system is to oscillatory rather than steady behavior. We briefly considered a few finite arrays (numerical simulations) in that work.

When the number of coupled stochastic oscillators is finite rather than infinite, there is no longer perfect synchronization. The previously stable states now experience fluctuations. Fluctuations due to finite numbers lead to transitions between the different previously stable states in the multistable state regime, that is, the states become metastable. These fluctuations become stronger as the number of oscillators decreases. It is our goal in this paper to analyze fluctuations due to finite numbers of members of the array, in the particular context of our chosen model. While the context is thus specific, the points that it illustrates about the effects of finite numbers are quite general. We have found very little work in the literature on finite arrays of phase coupled oscillators of any kind that undergo phase transitions in the thermodynamic limit. In fact, the only references we have identified deal with Kuramoto model. Even here finite arrays are still considered to be an open problem [15].

In Sec. II we introduce our model, that is, a single two-state oscillator and the general form (without yet choosing a particular form) of the coupling among oscillators to be considered in this paper. We present the microscopic equation that describes the dynamics of our array of \(N\) oscillators. This is the equation we use for most of our numerical simulations. We derive the associated large-\(N\) Langevin equation and from there the Fokker-Planck equation. As a parallel route we present the mesoscopic master equation for the array, which we also use in some of our numerical simulations. From this master equation we obtain the same Fokker-Planck equation when \(N\) is large. These various equations provide great clarity toward understanding the unanticipated effects of finite-\(N\) fluctuations. In Sec. III we choose particular coupling coefficients that lead to a pitchfork bifurcation when our array is infinite (as an aside, in the Appendix we solve the master equation for \(N = 2\) and show that even in this tiny array there are already signatures of the bifurcation). In Sec. IV we present a collection of results based on our various equations (dynamical equations, the master equation, and the Fokker-Planck equation) and ascertain the finite-\(N\) behavior in a variety of different ways. In Sec. V we summarize our results.

II. MODEL

We consider binary oscillators, that is, oscillators that can be in one of two states. These individual oscillators are stochastic because transitions between the two states of a binary oscillator occur according to given rates, that is, according to exponential distributions of transition times whose means are the inverses of these rates. Our model is thus Markovian. Later we introduce the second source of stochasticity, the one of particular interest in this study: Stochasticity also arises because we will consider arrays made up of a finite number \(N\) of these oscillators.

A. Single oscillator

We start with a single binary oscillator \(N = 1\) that can be either in state 0 or in state 1. The oscillator undergoes transitions from state 0 to state 1 at a rate \(\gamma_0\) and from state 1 to state 0 at a rate \(\gamma_1\), as illustrated schematically in Fig. 1. The probability \(p_1(t)\) of finding the oscillator in state 1 at time \(t\) is governed by the master equation

\[
\frac{dp_1(t)}{dt} = \gamma_0 p_0(t) - \gamma_1 p_1(t)
\]

where we used the normalization condition \(p_0(t) + p_1(t) = 1\). The stationary solution of this master equation is given by

\[
p_1^* = \frac{\gamma_0}{\gamma_0 + \gamma_1}.
\]

The decay of the probability \(p_1(t)\) to the stationary solution is found by solving the master equation and is exponential,

\[
p_1(t) = p_1^*(1 - e^{-(\gamma_0 + \gamma_1)t}) + p_1(0)e^{-(\gamma_0 + \gamma_1)t}.
\]

B. \(N\) uncoupled oscillators

If we have any number \(N\) of uncoupled oscillators, then the probability \(P(N_1,t)\) that \(N_1\) oscillators are in state 1 at time \(t\) is simply given by the binomial theorem

\[
P(N_1,t) = \frac{N!}{(N-N_1)!N_1!} p_1(t)^{N_1}[1 - p_1(t)]^{N-N_1}.
\]

The average fraction of oscillators in state 1, \(\langle n_1(t)\rangle = \langle N_1(t)/N\rangle\), equals to \((1/N)\sum_{N_1=0}^{N} N_1 P(N_1,t)\) and satisfies Eq. (1),

\[
\frac{d\langle n_1(t)\rangle}{dt} = \gamma_0 - (\gamma_0 + \gamma_1)\langle n_1(t)\rangle.
\]

The width of the distribution \(P(N_1,t)\) narrows as \(1/\sqrt{N}\), so if \(N \to \infty\), then the distribution about this average is infinitely narrow \(P(N_1,t) \to \delta(N_1 - \langle N_1(t)\rangle)\). In this limit we will dispense with the averaging brackets and simply write

\[
\frac{dn_1}{dt} = \gamma_0 - (\gamma_0 + \gamma_1)n_1(t).
\]

C. Coupled oscillators: Langevin equation

Here we begin our discussion of an array of \(N\) coupled oscillators. As with an array of uncoupled oscillators, each oscillator can be in either state 0 or state 1 and as time proceeds each oscillator can oscillate between these states. The system
at any given time has \(N_1(t)\) oscillators in state 1 and \(N - N_1(t)\) oscillators in state 0 and this number can change with time. The coupling among the oscillators occurs by way of the transition rates: The rates \(\gamma_0\) and \(\gamma_1\) now depend on the instantaneous state \(n_1(t) = N_1(t)/N\) of the system, \(\gamma_0(n_1)\) and \(\gamma_1(n_1)\). At any moment all oscillators are subject to the same transition rates, but these change as the state of the system changes.

Consider the evolution of the dynamical quantity \(N_1(t)\), the number of oscillators in state 1. This number is determined by the microscopic dynamics, that is,

\[
N_1(t + dt) = N_1(t) - \sum_{k=1}^{N} \theta(\gamma_1(n_1)dt - \zeta_k) + \sum_{k=N_1+1}^{N} \theta(\gamma_0(n_1)dt - \zeta_k), \tag{7}
\]

Here \(\theta(x)\) is the Heaviside step function, \(\theta(x) = 1\) if \(x > 0\), \(\theta(x) = 0\) if \(x < 0\), and \(\theta(0) = 1\) [or \(\theta(0) = 0\); this choice of the value at one point does not affect the results]. The set \(\{\zeta_k\}_{k=1}^{N}\) is a set of independent random variables uniformly distributed in the interval \([0, 1]\). Thus, if oscillator \(k\) is in state 1 at time \(t\), then if \(\gamma_1(n_1)dt > \zeta_k\), oscillator \(k\) flips from state 1 to state 0 and as a consequence \(N_1\) decreases by 1. If \(\gamma_0(n_1)dt < \zeta_k\), then oscillator \(k\) remains in state 1. On the other hand, if oscillator \(k\) is in state 0 at time \(t\), then it moves to state 1 if \(\gamma_0(n_1)dt > \zeta_k\), thus increasing \(N_1\) by 1. If \(\gamma_0(n_1)dt < \zeta_k\), then the oscillator remains in state 0. Therefore Eq. (7) is a straight counting protocol. Note that, at any given time \(t\), the dynamics of all oscillators in the array are determined by the same rates \(\gamma_0(n_1(t))\) and \(\gamma_1(n_1(t))\).

To further work with Eq. (7) it is thus necessary to analyze the random variables

\[
\phi_k = \theta(X_0 - \zeta_k), \quad \Phi_M = \sum_{k=1}^{M} \phi_k, \tag{8}
\]

where \(X_0 \in [0, 1]\) is a fixed quantity and \(M\) is an arbitrary integer between 1 and \(N\). The second quantity of these two counts how many oscillators out of \(M\) flip at time \(t\). Denoting the probability of event \(A\) by \(P(A)\), we have

\[
P(\phi_k = 1) = X_0,
\]

\[
P(\Phi_M = M') = \frac{M!}{(M-M')!(M')!} X_0^{M}(1-X_0)^{M-M'}.
\]

The first two moments of \(\Phi_M\) then give

\[
\langle \Phi_M \rangle = MX_0, \quad \sigma(\Phi_M) = \sqrt{\langle (\Phi_M - \langle \Phi_M \rangle)^2 \rangle} = \sqrt{MX_0(1-X_0)}. \tag{9}
\]

Moreover, for large \(M\)

\[
P(\Phi_M = M') \approx N(\langle \Phi_M \rangle, \sigma(\Phi_M)), \tag{11}
\]

where \(N(\mu, \sigma)\) denotes a Gaussian distribution of mean \(\mu\) and standard deviation \(\sigma\). This leads us to introduce the change of variables

\[
\Phi_M = \langle \Phi_M \rangle + \sigma(\Phi_M)\psi, \tag{12}
\]

where, for large \(M\), \(\psi \sim N(0,1)\).

Using this change of variables in Eq. (7) then gives

\[
\sum_{k=1}^{N_1} \theta(\gamma_1(n_1)dt - \zeta_k) = \frac{N_1\gamma_1(n_1)dt + \sqrt{N_1\gamma_1(n_1)dt} \psi_1,}{N_1 + \sum_{k=N_1+1}^{N} \theta(\gamma_0(n_1)dt - \zeta_k)} = (N - N_1)\gamma_0(n_1)dt + \sqrt{(N - N_1)\gamma_0(n_1)dt} \psi_0. \tag{13}
\]

where we have ignored terms of \(O(dt^2)\) inside the square roots as compared to terms of \(O(dt)\). We assume that the three quantities \(N, N_1,\) and \((N - N_1)\) are all much larger than unity, so that \(\psi_0\) and \(\psi_1 \sim \mathcal{N}(0,1)\). Furthermore, since the random variable \(\psi_1\) is generated from the set \(\{\zeta_k\}_{k=1}^{N_1}\), while \(\psi_0\) is generated from the set \(\{\zeta_k\}_{k=N_1+1}^{N}\), they are independent. Hence, if we introduce the random variable

\[
\psi = \frac{\sqrt{(N - N_1)\gamma_0(n_1)dt} - \sqrt{N_1\gamma_1(n_1)\psi_1}}{\sqrt{(N - N_1)\gamma_0(n_1)dt} + \sqrt{N_1\gamma_1(n_1)\psi_1}}, \tag{14}
\]

then also \(\psi \sim \mathcal{N}(0,1)\). Equation (7) now takes the form

\[
dN_1 = \gamma_0(n_1)(N - N_1)dt - \gamma_1(n_1)N_1dt + \sqrt{\gamma_0(n_1)(N - N_1) + \gamma_1(n_1)N_1} \overline{dt} \psi, \tag{15}
\]

where \(dN_1 = N_1(t + dt) - N_1(t)\). Finally, we now use this equation for the density \(n_1(t)\) and define

\[
\xi(t) = \frac{\psi}{\sqrt{dt}}, \tag{16}
\]

which, assuming \(\psi \sim \mathcal{N}(0,1)\), corresponds to a Gaussian white noise with

\[
\langle \xi(t) \xi(t') \rangle = \delta(t - t'). \tag{17}
\]

We have thus arrived at our result for the Langevin equation that governs \(n_1(t)\):

\[
n_1 = \gamma_0(n_1) - \gamma_0(n_1) + \gamma_1(n_1)n_1 + \sqrt{(1 - \gamma_0(n_1) + \gamma_1(n_1)\xi(t)\sqrt{N}_1}. \tag{18}
\]

Note that in the thermodynamic limit \(N \rightarrow \infty\), Eq. (18) reduces to the mean field equation

\[
n_1 = \gamma_0(n_1) - \gamma_0(n_1) + \gamma_1(n_1)n_1, \tag{19}
\]

which is of the same form as Eq. (6) for uncoupled oscillators but now with transition rates that depend on the state of the system.

The Langevin equation (18), when interpreted using Itô calculus \([16,17]\), leads to the Fokker-Planck equation for the probability \(P(n_1, t)\) that the fraction \(N_1/N\) is equal to \(n_1\) at time \(t\),

\[
\frac{\partial}{\partial t} P(n_1, t) = -\frac{\partial}{\partial n_1} [F(n_1)P(n_1, t)] + \frac{1}{2N} \frac{\partial^2}{\partial n_1^2} [G(n_1)P(n_1, t)], \tag{20}
\]

052143-3
where
\[ F(n) = (1 - n)\gamma_0(n) - n\gamma_1(n), \]
\[ G(n) = (1 - n)\gamma_0(n) + n\gamma_1(n). \]
(21)

The stationary solution of the Fokker-Planck equation is
\[ P(n_1) = C_N \exp\left[\frac{2NH_d(n_1)}{G(n_1)}\right], \]
where the effective potential is given by
\[ U_d(n_1) = \int_0^{n_1} \frac{F(n)}{G(n)} \, dn \]
(23)
and \( C_N \) is the normalization constant that ensures that the integral of the probability is unity.

We note a source of eternal confusion in the literature, namely, the difference between the \( n \) integral of the probability is unity.

We wish to understand this signature, in particular, the fluctuations induced by finite numbers.

We begin by focusing on Eq. (19). Before choosing a particular model, we note that we must choose the two rates to be different. Indeed, if the two rates are equal \( \gamma_0(n_1) = \gamma_1(n_1) \equiv \gamma(n_1) \), then Eq. (19) reduces to \( \dot{n}_1 = -2\gamma(n_1)(n_1 - 1/2) \). This equation has the trivial equilibrium \( n_1^* = 1/2 \), which is always stable. If we perturb away from this solution \( n_1 = 1/2 + \delta n_1 \), the perturbation satisfies \( \delta n_1 = -2\gamma(1/2)\delta n_1 \). Since \( \gamma \) is always positive because it is a rate, the solution \( \delta n_1 \) of this equation goes to zero exponentially. Moreover, it is clearly not possible to have more equilibria than the symmetric one \( n_1^* = 1/2 \). Therefore, to observe more than one steady state we must allow the system to have different transition rates.

Generally speaking, Eq. (19) has the form
\[ \dot{n}_1 = F(n_1, \varepsilon), \]
(27)
where \( \varepsilon \) denotes a control parameter, which may be a set of numbers. For simplicity, we consider it to be a single real number. The choice of \( F \) determines whether the system undergoes a phase transition and, if so, the type of transition. We choose to work with a prototype of a mean field second-order phase transition, namely, a pitchfork bifurcation. Other choices (saddle node or transcritical) and combinations thereof can lead to other second-order transitions and also to first-order (discontinuous) transitions.

Figure 2 illustrates the behavior of the function \( F \) for a pitchfork bifurcation. In some range of values of the control parameter, say, \( \varepsilon < \varepsilon_c \), the system has a single steady state \( n_1^* = n_1^*(\varepsilon) \), which may depend on the value of the control parameter. For \( \varepsilon > \varepsilon_c \), \( n_1^*(\varepsilon) \) becomes unstable and two new stable equilibria appear, say, \( n_1^* = n_1^*(\varepsilon) \) and \( n_1^* = n_1^*(\varepsilon) \). Hence, at the critical point \( \varepsilon = \varepsilon_c \), \( F \) has a saddle point at \( n_1^*(\varepsilon_c) \), that is,
\[ \frac{\partial F}{\partial n_1} \bigg|_{\varepsilon_c} = 0, \quad \frac{\partial^2 F}{\partial n_1^2} \bigg|_{\varepsilon_c} = 0, \quad \frac{\partial^3 F}{\partial n_1^3} \bigg|_{\varepsilon_c} < 0, \]
(28)
where the subindex \( c \) denotes evaluation at \( \varepsilon = \varepsilon_c \).

In our earlier work [13] we selected a particular model out of many possibilities because it leads to a well-characterized
potential for $\varepsilon > \varepsilon_c$

Note that this equality is not the other where the same majority of oscillators is in state 1. This fixed point is unstable when $\varepsilon < 0$, that is, half the oscillators are in state 0 and the other half in state 1. This fixed point is unstable when $\varepsilon > 0$ and two new stable fixed points appear at $n_1 = 1/2 \pm \sqrt{\varepsilon}$, exactly as illustrated in Fig. 2(b). Equation (30) implies a relaxation dynamics dictated by the potential

$$\dot{n}_1 = \varepsilon(n_1 - 1/2) - (n_1 - 1/2)^3.$$  

(30)

Defining $u \equiv n_1 - 1/2$, we arrive at the familiar form that motivated our choice, $\dot{u} = \varepsilon u - u^3$. The pitchfork bifurcation occurs at $\varepsilon_c = 0$. We have a stable fixed point at $n_1 = 1/2$ for $\varepsilon < 0$, that is, half the oscillators are in state 0 and the other in state 1. This fixed point is unstable when $\varepsilon > 0$ and two new stable fixed points appear at $n_1 = 1/2 \pm \sqrt{\varepsilon}$, exactly as illustrated in Fig. 2(b). Equation (30) implies a relaxation dynamics dictated by the potential

$$\dot{u}(n_1) = \frac{1}{2}[\varepsilon - (n_1 - 1/2)^3]^2,$$

(31)

in terms of which the mean field equations for $n_1$ can be written as $\dot{n}_1 = -\partial U/\partial n_1$. Figure 3 displays the typical shape of the potential for $\varepsilon > 0$, indicating two steady states of the same energy, one where the majority of oscillators is in state 0 and the other where the same majority of oscillators is in state 1. Note that this equality is not $a \text{ priori}$ evident since the rates $\gamma_0$ and $\gamma_1$ are different from each other.

**IV. RESULTS**

In this section we collect a number of results, both numerical and analytical, in our effort to unravel the effects of the fluctuations introduced by a finite rather than an infinite number of oscillators in our array. Some of our numerical results are based on the implementation of the microscopic dynamics as given in Eq. (7) and others on the implementation of the mesoscopic description Eq. (24). The results obtained from the Fokker-Planck equation are analytic (except for quadrature).

In the mean field limit $N \to \infty$, our array of globally coupled two-state Markovian oscillators evolves as described by Eq. (30). The evolution is a relaxation process dictated by the potential (31) and schematically shown in Fig. 3 when $\varepsilon > 0$. The populations in the two stationary states are equal because of the symmetry of the potential. When $\varepsilon < 0$ the potential is monostable, with half the oscillators in state 0 and the other half in state 1.

In the next set of figures, time $t$ is sufficiently long for the arrays to have reached a steady state. In Figs. 4 and 5 we show a number of distributions for finite arrays at two fixed values of $\varepsilon$. The discrete symbols in each case are the results of direct numerical simulations using the microscopic equations (7) and the lines are the solutions of the Fokker-Planck equation. The agreement between the two is gratifying in both cases.

Figure 4 is in the monomodal regime. For this value of $\varepsilon$, the distribution (22) exhibits one maximum: For the value $\varepsilon = 0.001$ used in the figures bimodality does not appear until $N \sim 2 \times 10^4$. The maximum is not at $1/2$ because the fluctuations due to finite $N$ are not symmetric (see below), the asymmetry growing stronger with decreasing $N$. This is why the maximum moves to lower values of $n_1$ with decreasing $N$. The increasing

$$P(n_1) = \frac{1}{\sqrt{2\pi \varepsilon}} \exp\left(-\frac{1}{2\varepsilon}(n_1 - 1/2)^2\right),$$

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(22)
width of the distribution is a manifestation of the increasing amplitude of fluctuations as \( N \) decreases.

Bimodality is clear in Fig. 5, where the distributions for the same values of \( N \) are shown, but now for \( \epsilon = 0.1 \). Here the maxima are slightly closer to the mean field values \( 1/2 \pm \sqrt{\epsilon} \) for the larger \( N \), but move away from these values with decreasing \( N \). The asymmetry of the fluctuations is clearly evident in the large difference between the values of the maxima. The normalization of the distributions must be considered when further analyzing the detailed trends in these figures.

The transition between the monomodal and bimodal distributions for a particular value of \( N \) is seen with great clarity in Fig. 6, where we show the probability of having \( N_1 \) oscillators in state 1 for different values of \( \epsilon \) in an array of \( N = 50 \) oscillators. The results in this figure have been obtained by numerical integration of the master equation (24). We conclude from Fig. 6 that for negative values of \( \epsilon \) the oscillators are more or less equally distributed in the two states, i.e., the maximum probability has 25 oscillators in each state, consistent with a potential with a single minimum, but now with a spread around these maxima. At some point with increasing \( \epsilon \), the distribution starts to develop an asymmetry; the value of \( \epsilon \) where this becomes apparent depends on \( N \) and lies between \( \epsilon = -0.05 \) and 0.05. For increasingly positive values of \( \epsilon \) we see two distinct but asymmetric maxima, consistent with a double-well potential.

Returning to Fig. 5, the first peak becomes higher and narrower as \( N \) increases. The second peak decreases as \( N \) increases and likely disappears as \( N \rightarrow \infty \). This appears contradictory to the mean field result discussed earlier, where the double-peaked distributions have peaks of equal heights (cf. Fig. 3). However, there is no inconsistency in these results: The difference arises from the fact that we have taken two limits that do not commute and the order of the limits has been reversed. In the mean field case we have taken the limit \( N \rightarrow \infty \) first and then the limit \( t \rightarrow \infty \). In our numerical results for finite \( N \) we allow the system to reach a steady state first, that is, we take the long-time limit first and then the large-\( N \) limit. The Fokker-Planck analysis is based mainly on the steady-state result (22), where we have explicitly taken the limit \( t \rightarrow \infty \) while \( N \) is still finite.

The source of the asymmetry in the finite \( N \) case is clearly seen in the Langevin equation (18) and also in the Fokker-Planck equation (20): The fluctuations caused by the finite total population depend on the state of the system. In the bistable regime the fluctuations that drive the system to a state with a small number of oscillators in state 1 dominate those that drive the system to a large number of oscillators in this state. This is illustrated in Fig. 7. Therefore, while there is still a bistable regime evidenced by the two maxima in the steady-state distribution, they are no longer symmetric as they are in the mean field limit when there are no fluctuations. This somewhat unanticipated behavior has been completely clarified by the analytic derivations. We can in fact give a quantitative value to the difference in the lower and upper peaks in the bimodal case: The difference diverges exponentially. If we denote the value of the curve at the first peak as \( P(\eta_1) \) and at the second peak as \( P(\eta_2) \), then

\[
\frac{P(\eta_1)}{P(\eta_2)} \sim \exp[2N(U_{\text{eff}}(\eta_1) - U_{\text{eff}}(\eta_2))] \quad \text{as} \quad N \rightarrow \infty .
\]  

In Fig. 8 we show the \((\epsilon, N)\) pairs of values at which the transition occurs from unimodal to bimodal behavior, using
two calculations. The unimodal regime lies to the left, the bimodal to the right. The closed circles are obtained from the numerical integration of the master equation (24) and the line is the prediction of the transition according to the Fokker-Planck equation. In the case of the Fokker-Planck equation the values can be found analytically from the stationary distribution (22). If we define the function

\[ f(\varepsilon, N) = 27[N^2(4\varepsilon - 1) + 6N + 8]^2 - 8[2N^2\varepsilon + 3N + 6]^2, \]

(33)

the Fokker-Planck prediction is that the distribution is monomodal when \( f(\varepsilon, N) > 0 \) and bimodal when \( f(\varepsilon, N) < 0 \). The transition thus occurs when

\[ f(\varepsilon, N) = 0. \]

(34)

For the master equation these points are found numerically. The agreement between the two is excellent, even in the small-\( N \) regime where the Fokker-Planck equation might not be expected to yield such accurate results. Exact results for \( N = 2 \) are found in the Appendix and are shown to already display features of the transition. Note that as \( N \) increases in the figure, both the points and the line cross slightly into the positive \( \varepsilon \) regime. This is a "real" result and not the outcome of inaccuracies in the calculations. We have ascertained that as \( N \) increases further, both calculations return the transition value toward \( \varepsilon = 0 \), the correct value as \( N \to \infty \).

In Fig. 9 we show a plot of \( \log |\varepsilon| \) vs \( \log N \) at the transition obtained from the Fokker-Planck equation. Finite-size scaling theory predicts that \( \varepsilon_c(N) - \varepsilon_c(\infty) = \text{const} \times N^{-1/\alpha} \), that is, that the critical value of the control parameter converges to its mean field value following a power law. Expanding Eq. (34) for large \( N \gg 1 \) leads to the following relation between \( \varepsilon \) and \( N \) at the transition:

\[ \varepsilon_c(N) \cong \frac{3}{4} N^{-2/3}, \]

(35)

that is, to \( \alpha = 3/2 = 1.5 \). The slope found in Fig. 9 gives \( \alpha = 1/0.64 = 1.56 \). The agreement between the numerical and analytic values is thus excellent.

We have at this point fully described the transition of the probability distributions of our finite arrays from ones in which the preferred configuration has as many oscillators in state 0 as in state 1 (monostable regime) to distributions where there are more oscillators in one state than in the other. We have also explained why the finite-\( N \) fluctuations lead to an asymmetry in the latter regime such that the distribution in which there are more oscillators in state 0 than in state 1 is more likely than the distribution in which there are more oscillators in state 1 than in state 0. In an infinite array there is no such asymmetry and we have also explained this difference. Both the monostable and the bistable distributions have widths that increase with decreasing \( N \) around the preferred state. Furthermore, the asymmetry in the bistable regime increases with increasing \( N \). Repeating what we said earlier, we showed that the fluctuations caused by the finite total population of the array depend on the state of the system and it is this dependence that leads to the asymmetry in the bistable regime. It is interesting to study this further by observing the actual time evolution of finite arrays, which we now proceed to do. We emphasize that even though the various distributions that we have discussed reach a stationary state and it is this state that we have focused on, each oscillator in the array, no matter the state or the size of the array, is transitioning back and forth between states 0 and 1 as time proceeds.

Figure 10 shows long-time evolution snapshots of arrays of various sizes in the bistable regime. The snapshots clearly illustrate the fact that the lower metastable state (the one with fewer oscillators in state 1 than in state 0) is much more stable and highly probable than the upper metastable state. The fluctuations around the lower state are seen for all array sizes tested. The larger arrays remain in that region for the duration of this simulation. We see transitions to the upper state and fluctuations around it prior to a return to the lower state for the smaller arrays. The intermediate-sized arrays show an occasional attempt toward the upper state (or perhaps just a large fluctuation around the lower state). In any case, this figure...
clearly exhibits the asymmetry between the two metastable states due to the differences in the fluctuations noted earlier.

Finally, to support and reinforce some of the results presented so far, we show results for the passage from one of the metastable states to the other in the bimodal phase. Recall that in an infinite array these states are both stable, sharply defined, and equally probable. In a finite array there is now a distribution of possible states, with maxima at two metastable states. The fluctuations due to finite numbers can cause a transition from a state around one of the maxima to a state around the other. Since the fluctuations break the symmetry of the infinite array, we expect an asymmetry in the rates that we have chosen leads to a pitchfork bifurcation of the system as a function of a control parameter that appears in the rates. In this (mean field) limit the stable state below the bifurcation is one with an equal number of oscillators in each of the two states; above the bifurcation there are two symmetric stable states, one with a preponderance of oscillators in one of the two states and the other with the same preponderance of oscillators in the other state. We determined this number is arbitrary, it ensures that we do not miscount events where an oscillator appears to exit the vicinity of one metastable state but returns there before reaching the other vicinity. The particular choice does not matter, but we have ascertained that with this choice, the number of recrossings is exceedingly low. We expect the fluctuation-induced transitions from the vicinity of the lower to that of the upper maximum seen, for instance, in Fig. 5, to occur less frequently than those that take the system from around the upper to around the lower maximum. Figure 11 clearly illustrates these results.

While the distribution of first passage times confirms our picture, the quantity that is more often used as a measure of this transition is the mean first passage time [18]. If we start our system with a fraction \( \eta_0 \) of the oscillators in state 1, that is, in state \( P(n,t) = 0 = \delta(n - \eta_0) \), we can ask how long it takes on average for the array to reach the state in which \( \eta \) of the oscillators are in state 1 for the first time. If \( \eta_0 < \eta \), the average value of this time takes the form

\[
\langle t(\eta_0|\eta) \rangle = 2N \int_{\eta_0}^{\eta} \frac{dn}{\bar{G}(n)} P(n) \int_{n}^{\infty} P(n') dn'.
\]  

(36)

If \( \eta_0 > \eta \),

\[
\langle t(\eta_0|\eta) \rangle = 2N \int_{\eta_0}^{\eta} \frac{dn}{\bar{G}(n)} P(n) \int_{n}^{\infty} P(n') dn'.
\]  

(37)

Here \( P(n) \) is the stationary-state distribution given in Eq. (22). In Fig. 12(a) we show the results for the mean first passage times from the state with \( n_1 = 0 \) (the lowest value in the lower attractor) to the upper attractor for an array of \( N = 100 \) units. In Fig. 12(b) we show the results for the mean first passage times from the state with \( n_1 = 1 \) (the highest value in the upper attractor) to the lower attractor for the same array. The dotted curves are the numerical results obtained from Eq. (7) and the solid lines are the results of Eq. (36) for Fig. 12(a) and Eq. (37) for Fig. 12(b). The Fokker-Planck equation is seen to produce quite accurate results. The two panels clearly support the longer residence time in the lower attractor.

V. CONCLUSION

We have analyzed finite arrays of coupled two-state (i.e., on and off) oscillators coupled together via the dependence of the transition rates between the two states for each oscillator on the global time-dependent state of the system. In the thermodynamic limit, the global state dependence of the transition rates that we have chosen leads to a pitchfork bifurcation of the system as a function of a control parameter that appears in the rates. In this (mean field) limit the stable state below the bifurcation is one with an equal number of oscillators in each of the two states; above the bifurcation there are two symmetric stable states, one with a preponderance of oscillators in one of the two states and the other with the same preponderance of oscillators in the other state. We determined the effective potential for the mean field evolution and showed that, in accordance with the above results, this potential is
either symmetrically monostable or symmetrically bistable, depending on the value of the control parameter.

Our work has focused on the fluctuations that occur naturally when the number $N$ of oscillators in our array is finite rather than infinite. These fluctuations of course become stronger as the number of oscillators in the array decreases. We derived a Fokker-Planck equation for the array in two ways, one starting from a set of microscopic dynamical equations from which we derive a Langevin equation and the other from a mesoscopic master equation. We analyzed the behavior of the system in the stationary state on the basis of the direct numerical simulation of the microscopic equations, from an integration of the master equation, and from the Fokker-Planck equation. We showed that a transition as a function of the control parameter still takes place, now from a monomodal distribution to a bimodal distribution at a value of the parameter depending on $N$. We identified a somewhat unanticipated behavior in the observation that the peak heights and widths of the bimodal distributions are not symmetric, although the peak heights are symmetric in the thermodynamic limit. The asymmetry is introduced by the fluctuations, which we show to be multiplicative (in the Langevin formulation). We again determine an effective potential, which is no longer symmetric. The asymmetry of the fluctuations causes one of the minima in the bistable regime to be deeper than the other. This difference grows as the number of oscillators $N$ increases. Care must be exercised when taking the thermodynamic limit: If time $t$ is taken to infinity before $N$, then this difference continues to grow until one of the maxima effectively disappears. If $N$ is taken to infinity first, then the result is the mean field symmetric model.

The results obtained from the Fokker-Planck equation, from direct integration of the master equation, and from the direct simulation of the microscopic equations agree extremely well with one another. In addition to the stationary state, we also presented results for time-dependent behavior. The value of these additional results lies in the further clarification of the effect of the asymmetry introduced by the fluctuations.

Efforts arising from this and previous work continue; among them are studies of coupled stochastic two-state finite arrays such as those discussed in this paper, but leading to a discontinuous transition [19]. Another is an analysis of arrays of three-state coupled stochastic units similar to those studied in our earlier work [7–10], but now with negative coupling parameters [20]. This leads to behavior entirely different from that found before. Another is a study of noise-induced transitions in arrays of non-Markovian discrete state units [14]. Yet another is the study of the effect of finite-number-induced fluctuations on oscillatory three-state models [21]. It is quite surprising how much information about synchronization phenomena is yet to be learned from these relatively simple discrete-state systems.

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**APPENDIX: TWO OSCILLATORS**

The problem of two oscillators can be solved analytically in full generality. There are, at any time, three possible situations, denoted by the probabilities $P(N_1,t)$, where $N_1$ is the number of oscillators in state 1: Neither oscillator is in state 1, with probability $P(0,t)$; one oscillator is in state 1 and the other in state 0, with probability $P(1,t)$; and both are in state 1, with probability $P(2,t)$. These three probabilities add up to unity. This is the same function appearing in Eq. (24). For this case we can solve the master equation exactly to obtain the eigenvector-eigenvalue $V_\lambda$ decomposition

$$P(N_1,t) = \sum_\lambda C_\lambda e^{\lambda t} V_\lambda,$$

where $\lambda_0 = 0$ is the eigenvalue associated with the stationary solution $V_0$, $\lambda_1 = -\frac{1}{2}f_1 + \sqrt{f_1^2 - f_2^2}/2$, and $\lambda_2 = -f_1/2 - \sqrt{f_1^2 - f_2^2}$, with

$$f_1 = \gamma_0(\frac{1}{2}) + \gamma(\frac{1}{2}) + 2[\gamma_0(0) + \gamma_1(1)],$$

$$f_2 = 8[\gamma_0(0)\gamma_0(\frac{1}{2}) + \gamma_1(\frac{1}{2})\gamma_2(1) + 2\gamma_0(0)\gamma_1(1)].$$
The expansion coefficient $C_0$ is given by

$$C_0 = \frac{\gamma_0(0)\gamma_1(1)}{\gamma_0(0)\gamma_1(1/2) + \gamma_1(1/2)\gamma_2(1) + 2\gamma_0(1)\gamma_1(1)}$$  \hspace{1cm} (A3)$$

and the other two coefficients are determined by the initial condition via $C_1\lambda_1 + C_2\lambda_2 = \gamma_0(1/2)P(1,0) - 2\gamma_1(1)P(1,0) + \gamma_2(1)P(1,0) = 1$. Note that if the two oscillators are uncoupled, the rates $\gamma_0$ and $\gamma_1$ do not depend on the state of the system and the eigenvalues reduce to $\lambda_1 = -(\gamma_0 + \gamma_1)$ and $\lambda_2 = -2(\gamma_0 + \gamma_1)$. These are real, so the system approaches the steady state exponentially. However, if the oscillators are coupled, then the system exhibits an oscillatory approach toward equilibrium if $f_2 > f_1^2$. Without a specific model for the coupling, this is as far as we can go in the analysis of two oscillators.

The coupling coefficients (29) for $N = 2$ are

$$\gamma_0(0) = \gamma_1(1) = \frac{1}{8} - \frac{\varepsilon}{2}, \hspace{1cm} \gamma_0(1/2) = \gamma_1(1/2) = \frac{1}{2} - \frac{\varepsilon}{2}.$$  \hspace{1cm} (A4)$$

With this choice, the eigenvalues are $\lambda_1 = -1 + 5\varepsilon/2$ and $\lambda_2 = -4 + 7\varepsilon/2$. The system approaches its steady state exponentially, at rate $\lambda_1$, and the steady state has probabilities for the three states given by

$$P(0) = P(2) = \frac{2(1 - \varepsilon)}{5 - 8\varepsilon}, \hspace{1cm} P(1) = \frac{1 - 4\varepsilon}{5 - 8\varepsilon}.$$  \hspace{1cm} (AS)$$

Thus the probabilities for both oscillators to be in state 0 or in state 1 are the same and the probability for each oscillator to be in a different state is larger than the former when $\varepsilon < -0.5$ and smaller when $\varepsilon > -0.5$ all the way to 0.25. At $\varepsilon = 0.25$ the probability that the oscillators are in different states vanishes and the probability that they are in the same state goes to unity. This progression can be seen in Fig. 13. What this figure illustrates in a rudimentary manner is a transition of an underlying effective potential from a single well to a double well. When $\varepsilon < -0.5$ the system can be thought of as being in an effective potential with a minimum at $N_1 = 1$ and therefore state 1 is the most likely state. The opposite is the case when $\varepsilon > -0.5$.

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