PulsePropagation in Granular Chains
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Abstract. We study pulse propagation in one-dimensional granular chains of spherical granules. Our goal is to develop an analytic approach that makes it possible to predict pulse properties as a function of time. Our method is based on a binary collision approximation that reduces the problem to collisions involving only two granules at a time. We illustrate the procedure and show quantitative agreement with exact numerical simulations for chains with smoothly varying granular profiles. For some more complex configurations (specifically, for decorated chains) we first replace the actual chain with a chain of smooth profile with renormalized masses and intergranular interactions. The binary collision approximation is then applied to this effective chain. Quantitative agreement with numerical simulations of the original chain is again shown. Further applications of the approach are briefly mentioned.

Keywords: Granular Chains
PACS: 05.45.-a,45.70.Vn

INTRODUCTION

Pulse propagation in granular media has spawned a great amount of work that ranges from the experimental to the numerical and analytical. Granular media exist in nature, for example in the large variety of sand beds and other sedimentary systems, and also in man-made forms such as shock absorbers and bulletproof vests. Man-made granular materials are frequently used to dampen the effect of a shock or energy pulse, and in these systems the goal is to optimize the way the system disperses and/or dissipates energy [2, 14]. Other potential applications are designed with the opposite goal in mind, for example that of focusing to generate high-energy acoustic pulses [31]. In natural granular media one might wish to use a pulse to detect a buried object, in which case it is again desirable for the pulse to remain focused as it penetrates the granular bed and bounces back [32].

One-dimensional granular chains have been a particular focus of experimental, numerical, and analytical studies. While such chains are at best model systems for more realistic higher dimensional configurations, they are already sufficiently complex to provide an interesting testbed, and with them one can for the moment avoid very difficult questions involving the transfer of energy between translational and rotational motion. Analytic studies have essentially focused entirely on one-dimensional granular chains. Even with this simplification, there are already a large number of issues that can be
addressed concerning pulse propagation. For example, what sort of chain configuration (sequence of sizes and masses) would optimize the shock absorption capability of a granular chain? What is the effect of gravity or other precompression state on the performance? How does the length of the chain affect the result? How about the shapes of the granules? What difference does it make if the granules are in a dry or a wet environment? The questions are endless and interesting. Beyond this, one can ask the very same range of questions if instead of imparting an energy pulse one excites the chain in a different way.

Here we review some recent progress in our understanding of pulse propagation in granular chains consisting of spherical granules. We shall discuss some numerical results and then show how to model such systems to obtain analytical insights into the pulse propagation in different kinds of chains.

THE MODEL

The interaction between two granules depends mainly on their shapes and on the overlap between them. In almost all the cases where the overlap between granules is small compared to their dimensions, the interaction potential can be modeled via the Hertz potential [12]

\[ V = \frac{a}{n} r'_k (y_k - y_{k+1})^n \theta(y_k - y_{k+1}). \]  

(1)

Here \( y_k(\tau) \) represents the displacement at time \( \tau \) of the center of mass of the \( k \)th granule from its starting position, \( a \) depends on the properties of the material of the colliding granules, and \( n \) depends on the shape of the granules. For spheres \( n = 3/2 \), and for side-to-side cylinders \( n = 2 \). For some geometries the interaction may involve sums of contributions of the form (1) with different \( n \)'s in each term [11, 21]. We have defined \( r'_k = [2R'_k R'_{k+1}/(R'_k + R'_{k+1})]^{1/2} \), where \( R'_k \) is the principal radius of curvature of the \( k \)th granule at the point of contact with granule \( k + 1 \). Note that when granules \( k \) and \( k + 1 \) do not overlap, the Heaviside function \( \theta \) vanishes, as does the interaction. It is this feature that differentiates this system from the more usual anharmonic chain in which there are attractive restoring forces. Here, if two granules find themselves moving in opposite directions, the chain simply fragments at that location.

We will introduce some restrictions at the outset, starting with our focus on one-dimensional chains of spherical granules. We assume all granules to have the same density \( \rho \) and so the mass of the \( k \)th granule is \( M_k = (4/3) \pi \rho (R'_k)^3 \). Initially all the granules are placed along the line such that they just touch their neighbors but with no overlap between them. We impart an initial velocity impulse \( V_1 \) to the left-most granule, \( k = 1 \), which causes it to collide with its neighbor \( k = 2 \). It proves convenient to introduce the dimensionless parameter \( \alpha = M_1 V_1^2/a(R'_1)^{n+1/2} \). The time evolution of the displacement of the \( k \)th granule is then obtained from the solution of the Newton’s equation of motion,

\[ m_k \frac{d^2 x_k}{dt^2} = r_{k-1} \delta^{n-1}_{k,k-1} - r_k \delta^{n-1}_{k,k+1}, \]  

(2)
where \( \delta_{kk'}^n \equiv (x_k - x_{k'})^n \theta(x_k - x_{k'}'). \) We have introduced the scaled variables \( x_k = y_k (\alpha^{-1/n}/R'_1) \), \( t = \tau(V_1 \alpha^{-1/n}/R'_1) \). We have also introduced the rescaled parameters \( m_k = M_k/M_1 \), \( r_k = [2R_k R_{k+1}/(R_k + R_{k+1})]^{1/2} \), and \( R_k = R'_k/R_1 \). With these changes, the initial velocity of the first granule is unity.

The solution of this coupled set of equations of motion is difficult if not impossible in view of the nonlinearity of the potential. Our recourse is to solve the equations numerically for a finite number of granules, or to implement approximations that allow analytic solution. We have implemented such an approximation and will present some of its successful predictions. We point out that for smooth chain geometries (e.g. monodisperse chains, or systematically tapered chains) the initial pulse tends to travel down the chain as a narrow pulse or at the least with a narrow front. It is this feature that has been exploited in most of the analytic work.

**ANALYTIC APPROACHES**

Two main analytic approaches have been implemented for this problem. One was first formulated over two decades ago by Nesterenko [18, 16, 19] and is based on a Taylor series expansion of \( x_{k \pm 1} \) around \( x_k \) in the equation of motion. The resulting continuum equation captures discreteness effects by going to fourth order in the expansion. The original Nesterenko solution was later extended to different geometries, and various constants that appear in the solution were later determined first numerically [13, 9, 10] and then analytically [22]. For a chain of monodisperse spheres the solution is

\[
\left( -\frac{\partial x}{\partial k} \right) = - \left( \frac{5c_0^2}{4} \right)^2 \sin^4 \sqrt{\frac{2}{5}} (k - c_0 t) \tag{3}
\]

with \( c_0 = 0.836 \). The pulse retains practically all (99.7%) of the initial energy, and there is no backscattering or fragmentation. The velocity depends on the amplitude, and the pulse simply moves without spreading or growing. The pulse resides on about five granules, with three carrying most of the amplitude.

An approximation entirely different in spirit and better suited to chains with variable radii is based on the notion that if the pulse or pulse front is sufficiently narrow, it might be adequate to assume that the pulse travels along the chain via a sequence of successive binary collisions, that is, that the main “action” at any one time resides on two granules [23, 24, 6, 7, 8, 21]. In more detail, within this binary collision approximation (BCA) the transfer of energy from granule \( k \) to granule \( k+2 \) is assumed to involve two sequential processes. In the first instance, granule \( k \) interacts with granule \( k+1 \) and transfers its energy to it while granule \( k+2 \) is assumed to be at rest. After a time \( T_k \) when the transfer of energy from \( k \) to \( k+1 \) is completed, granule \( k+1 \) starts to interact with granule \( k+2 \) and this process is repeated along the chain. This approximation becomes more and more reliable as \( n \) increases and is exact for a hard-sphere system. More explicitly, in the binary collision approximation the coupled equations of motion are broken down into the pair

\[
m_k \ddot{x}_k = -r_k \delta_{k,k+1}^{3/2}, \quad m_{k+1} \ddot{x}_{k+1} = r_k \delta_{k,k+1}^{3/2} \tag{4}
\]
which are easily solved recursively.

Implementing conservation of energy and momentum, the velocity of the \((k+1)\)th granule is found to be related to that of the \(k\)th granule as \(v_{k+1} = v_k/(1 + m_{k+1}/m_k)\), where \(v_k\) is the maximum velocity reached by granule \(k\) as the granule travels along the chain. The recursive relation can be solved iteratively to obtain

\[
v_k = \prod_{k' = 1}^{k-1} \frac{2}{1 + \frac{m_{k'+1}}{m_k}}.
\]  

(5)

The BCA also allows us to find an analytic expression for the time \(t_k\) that it takes the pulse to reach a particular granule in the chain. This is the sum of times taken by each granule to pass the energy on to the next granule, \(t_k = \sum_{k' = 1}^{k} T_{k'}\). Note that in a hard sphere system \(T_k = t_k = 0\). To evaluate \(T_k\) we introduce the variable \(\eta_k = x_k - x_{k+1}\). Subtracting the two equations in (4) we then have

\[
\frac{d^2\eta_k}{dt^2} = -\frac{r_k}{\mu_k} \eta_k,
\]

where \(\mu_k\) is the reduced mass of the two granules, and the unwritten Heaviside function on the right side is understood. Equation (6) is actually the equation of motion of the center of mass of our two-granule system, which is equivalent to the dynamics of a particle of mass \(\mu_k\) in a potential \((r_k/n)\eta_k^2 \theta(\eta_k)\). Energy conservation then tells us that at any time during a collision between the two granules

\[
\frac{1}{2} \mu_k \dot{\eta}_k^2(t) + \frac{r_k}{n} \eta_k^2(t) = \frac{1}{2} \mu_k \dot{\eta}_k^2(0),
\]

where the term on the right of the equation is the initial kinetic energy while the left side represents the sum of kinetic and potential energies of the center of mass at time \(t\). Here \(t = 0\) is defined as the time at which the two particular particles \(k\) and \(k + 1\) have just touched each other. Since initially only the \(k\)th granule is moving, \(\dot{\eta}_k = v_k\). It then follows from (5) that \(\dot{\eta}_k(0) = v_k\). We next define the time \(T_k\) of energy transfer between \(k\) and \(k + 1\) as the time during which the velocity of the \((k+1)\)st granule increases from zero and becomes equal to that of the \(k\)th granule. Note that in a chain of only two granules this time is actually twice \(T_k\), but in a chain we have more than two granules, so we need to divide by 2 since the interaction of granule \((k + 1)\) with \((k + 2)\) is not negligible during the second half of the time. Within this approximation, the time of energy transfer \(T_k\) is exactly equal to the time at which the compression between the two granules reaches its maximum value \(\eta_k^\text{max}\). This value can be obtained from (7) once we realize that at the maximum compression both granules are moving with the same velocity so that \(\dot{\eta}_k = 0\). We then have

\[
\eta_k^\text{max} = \left(\frac{n \mu_k \dot{\xi}_k^2(0)}{2r_k}\right)^{1/n}.
\]

The time \(T_k\) is then obtained as

\[
T_k = \int_0^{\eta_k^\text{max}} \frac{d\dot{\xi}_k(t)}{\dot{\xi}_k(t)} = \sqrt{n} \left(\frac{n \mu_k}{2r_k}\right)^{1/n} \frac{\Gamma(1 + 1/n)}{\Gamma(1/n + 1/2)} \left[\dot{\xi}_k(0)\right]^{n-2},
\]

(8)

where we have solved (7) for \(\dot{\xi}_k(t)\). For a monodisperse chain of spherical granules this yields \(T \approx 1.2194\).
For a monodisperse chain the continuum approximation improves as the exponent \( n \) decreases and the traveling pulse becomes wider. The BCA approximation, on the other hand, improves when \( n \) increases and the pulse becomes narrower. There is a fortunate confluence for a chain of spherical granules: both approximations are excellent. The relative error in the predicted pulse velocity when compared to numerical simulations is less than 3% with either approximation [24]!

TAPERED CHAINS

The continuum approximation becomes less practical as we depart from the monodisperse condition. The BCA, on the other hand, is particularly well adapted to some such deviations, a few of which we discuss below. In particular, we consider so-called tapered chains in this section and decorated chains in the next, and compare our analytic results with numerical simulations of the full equations of motion. These particular chains have been chosen because they attenuate pulses much more effectively than do monodisperse chains and are therefore good candidates for shock absorption [3, 4, 6, 17, 15, 29, 30, 33].

In linearly tapered chains (TCs) the radii of successive granules either increase or decrease linearly along the chain according to \( R_k = 1 \pm S(k-1) \). The tapering parameter \( S \) is positive, and the upper and lower signs correspond to backward and forward TCs respectively. Note that in a forward TC of \( N \) granules, the value of \( S \) has an upper bound, \( S < 1/(N-1) \).

A large-\( k \) expansion of the resulting velocity product leads to the behaviors [6]

\[
v^B_k \sim k^{-3/2}, \quad v^F_k \sim \left(1 - \frac{k}{1+1/S}\right)^{-3/2},
\]

where the superscripts \( B \) and \( F \) here and henceforth are used to denote “backward” and “forward” respectively. The divergence in the forward chain lies beyond the constraint of the inequality mentioned above and thus does not appear in the physical regime. The pulse amplitude decays for the backward TC and grows for the forward TC, both with \( S \)-independent power law. However, it is straightforward to see that the energy \( \frac{1}{2} m_k v^2_k \) decreases for both chains, as desirable for shock absorption.

For large \( k \), a similar analysis for the time \( T_k \) connects the pulse amplitude \( v_k \) with real time \( t_k \). We find [6]

\[
t^B_k \sim v_k^{-23/15}, \quad t^F_k \sim C(S) \left[1 - (1+S)^{23/10} v_k^{-23/15}\right],
\]

where

\[
C(S) \approx \frac{10\sqrt{\pi}}{23} \left(\frac{5}{8}\right)^{2/5} \frac{\Gamma(7/5)}{\Gamma(9/10)} (1+1/S).
\]

The \( \approx \) sign indicates that we have ignored the higher order terms in \( S \), since in almost all cases of interest \( S \ll 1 \). The results in (10) can be inverted to obtain the \( t \) dependence of the pulse amplitude. This together with (5), (9) and (10) constitute the set of predictions for linear TCs of the BCA to be compared with exact numerical solutions.
Exponentially Tapered Chains

In exponentially TCs the radii of successive granules increase or decrease geometrically, \( R_k = (1 \mp q)^{k-1} \). For backward tapering \( 0 < q < 1 \), while \( q > 0 \) for forward tapering. The closed-form velocity solution valid for all \( k \) is here found to be [6]

\[
    v^B_k = A(q) e^{-k \log A(q)}, \quad v^F_k = A(-q) e^{-k \log A(-q)},
\]

where \( A(q) = \frac{1}{2} [1 + (1 - q)^{-3}] \). It is again straightforward to ascertain that for either tapering the pulse energy decreases along the chain.

As before, we can use these results to compute the time that it takes the pulse to reach a particular granule. We obtain

\[
    t^B_k \sim e^{k \eta(q)} \quad \text{and} \quad t^F_k \sim 1 - e^{(k-1) \eta(-q)},
\]

where we have defined \( \eta(q) = \log \left\{ [A(q)]^{1/5}/(1-q) \right\} \). The time to reach a granule in the chain depends on the speed \( c(k) \) of the pulse which can be obtained by differentiating \( v_k \) with respect to \( k \).

\[
    c^B(k) \sim \frac{1}{\eta(q)} e^{-k \eta(q)}, \quad c^F(k) \sim \frac{1}{|\eta(-q)|} e^{(k-1)|\eta(-q)|}.
\]

Note that for \( k - 1 \gg 1/|\eta(-q)| \), the propagation time of the pulse becomes a constant. This is associated with the exponential divergence of the pulse speed in (13).

Comparison With Numerical Results

We next present some numerical results for the solution of the exact equations of motion for comparison with results of the BCA. We present results only for the linearly TCs for lack of space. We refer the reader to Ref. [6] for the case of exponentially TCs.

In Fig. 1 we compare the numerical and analytical results for the pulse amplitude for linear TCs at different values of the tapering parameter. Note that for backward TCs, in some regimes of the figure, the decay of the pulse amplitude is S-dependent. The S-independent power-law decay (9) is only valid when \( Sk \gg 1 \), which is reached earlier for larger \( S \) values. Figure 1 reveals the single weakest aspect of the theory in that it shows substantial differences in the absolute magnitude of the pulse amplitude between the theoretical and numerical integration results. However, the rate of decay of \( v_k \) is captured extremely well, within 1\% - 2\%, as shown in the inset. The inset also shows the approach to the S-independent power \( 3/2 \) for large \( k \), as predicted in (9).

The numerical results for the decay of the pulse amplitude with time is shown in Fig 2. The inset shows the exponent of the power-law decay as obtained from the fit to the numerical data. The BCA prediction for this exponent is \( 15/23 \approx 0.652 \) [see Eq. (10)] and is within 2\% - 3\% of the numerical result. The change in the pulse amplitude with time for the forward TC as obtained from the BCA in (10) can be expressed as

\[
    v(t) \sim \left( 1 - \frac{1}{C(S)} \right)^{-15/23}.
\]

This form agrees well with the numerical results in Fig. 2. However the exact time at which \( v(t) \) diverges is not very well captured by the formula.
FIGURE 1. The numerical result for the pulse amplitude (solid circles) is compared with the binary collision approximation results (open circles). Left panel: decay for backward linearly TCs with \( S = 0.2, 0.4, 0.6, \) and 0.9, from top to bottom. The inset shows the decay exponent, whose theoretical value approaches \( 3/2 \). Right panel: increase for forward linearly TCs for \( S \) from 0.001 to 0.008 in steps of 0.001 (bottom to top). The inset shows the variation in the parameter \( b \) with \( S \) in the fit \( v_k = a(1 - bk)^{-3/2} \) and a comparison of it with the prediction from the theory: \( b = (1 + 1/S)^{-1} \), Eq. (9) with (10). Figures from Ref. [6].

FIGURE 2. Left panel: velocity pulse amplitude vs time for backward linearly TCs with \( S = 0.4 \) to 0.9 in steps of 0.1 from top to bottom, as obtained from numerical integration. The inset shows the rate of decay for different \( S \) values. The long-time binary prediction is \( 15/23 = 0.652 \). Right panel: \( v \) vs \( t \) for forward linearly TCs with \( S = 0.001 \) to 0.01 in steps of 0.001 from bottom to top. Figures from Ref. [6].

This is due to the discrepancy in the absolute value of the pulse amplitude. Apart from that, the divergence exponent \(-15/23\) predicted by the theory is quite accurate.

Finally, we compare the results for the pulse residence time \( T_k \) on granule \( k \). Figure 3 shows that the results from the BCA are in excellent agreement with the numerical data - indeed, the two data sets are barely distinguishable in the figures.

We have thus seen that a simple binary collision approach can be successfully used to study pulse propagation in smoothly tapered chains. Our next question then is, can the BCA approach be applied to more general situations where the profile variations of successive granules are not smooth but are instead more abrupt? We deal with some such situations in the next section.

**PULSE PROPAGATION IN DECORATED CHAINS**

A decorated chain consists of large granules separated by (“decorated” with) smaller granules in between [3, 7, 28]. It has been found in numerical simulations [28] that the shock absorption properties of a granular chain can be greatly enhanced by decorating
FIGURE 3. The residence time $T_k$ of the pulse on granule $k$ for linearly TCs. Results from the BCA (open circles, barely visible because of the agreement between theory and numerical integration) and direct numerical integration (solid circles) are shown for different tapering parameters. Left panel: backward TC, bottom to top, $S=0.1$ to 0.9 in steps of 0.1. Right panel: forward TC, top to bottom, $S=0.001, 0.002, 0.003, 0.005$ and 0.009. Figures from Ref. [6]

FIGURE 4. Schematic of a simple decorated granular chain and the associated effective chain.

it. A schematic of a decorated chain is shown in the left panel of Fig. 4. In this schematic the large granules are monodisperse, as are the small granules.

It is clear that the BCA as formulated so far can not be applied to this chain. As a pulse travels along, because of the large size/mass difference of consecutive granules, each small granule will rattle back and forth several times as one large granule transmits its energy onto the next. The collisions therefore involve at least three particles rather than two, and the small particle undergoes oscillatory motion rather than the smooth motions that we have seen for the large granules in our smoother chains.

To overcome this problem, we recently developed a two-step approach [7]. The first step consists of “replacing” the decorated chain with an effective chain of only large renormalized masses and interactions that include the effects of the small granules in this renormalization. We then apply the BCA to the effective chain. Note that we always take the end granules to be large.

Effective Chain and Binary Collision Approximation

We briefly sketch the procedure to arrive at an effective chain [7]. This is most easily accomplished by focusing on a chain of five granules, three large ones and two small ones. This chain will be replaced with an effective chain of three large granules with renormalized masses and interactions. A minimum of five granules is necessary to capture the description of the effective end granules, which are somewhat different from interior granules [8].

We take the small granules to be monodisperse, a restriction that can be relaxed. The five granules are labeled as $k-2, k-1, k, k+1,$ and $k+2$. Granules $k-1$ and $k+1$ are the small ones of radius $r$ and the remainder are large ones of radii $R_i > r$, $i = k, k \pm 2$. 
The leftmost end granule is given a unit velocity impulse. As energy is transferred from one large granule to the next through the small one separating them, the small granule executes oscillations. For small granule \( i \) we write,

\[
x_i(t) = \bar{x}_i(t) + A \sin(\omega t + \phi).
\]

Here the amplitude \( A \), the frequency \( \omega \) and the phase \( \phi \) all depend on the size ratio between the big and the small granules and they are also functions of time. The small granule oscillates around its average displacement \( \bar{x}_i(t) \), which also changes with time as the small granule is constantly pushed by the bigger one on its left toward the rightmost end of the chain. However, during the time of transfer of energy between two big granules these quantities can be approximated as time independent. Equation (14) is substituted into the equations of motion of the small granules to eventually be eliminated from the picture.

We next write the equations of motion for the large granules with the substitution (14). We assume that the amplitude \( A \) of the oscillations is small compared to the difference \(|x_k - \bar{x}_{k-1}|\). We can then implement a Taylor expansion in \( A \). The lowest term in the expansion gives the equation for the average displacement of the small granule \((k - 1)\),

\[
m_{k-1} \ddot{\bar{x}}_{k-1} \approx r_{k-2} (x_{k-2} - \bar{x}_{k-1})^{n-1} - r_{k-1} (\bar{x}_{k-1} - x_k)^{n-1}.
\]

The next term connects the frequency \( \omega \) to the average displacement, \( m_{k-1} \omega^2 \approx r_{k-2} (x_{k-2} - \bar{x}_{k-1})^{n-2} + r_{k-1} (\bar{x}_{k-1} - x_k)^{n-2} \). This frequency attains its maximum value when the small particle is at its “equilibrium” position, that is, its position when the forces on either side of it cancel each other. Thus at the time when the small granule oscillates with maximum frequency we have \( \dot{\bar{x}}_{k-1} = 0 \). Applying these steps to either small granule \( i \) in the chain then finally gives

\[
\bar{x}_i(t) = \frac{\alpha_i x_{i+1}(t) + x_{i-1}(t)}{1 + \alpha_i}, \quad \alpha_i = \left( \frac{r_i}{r_{i-1}} \right)^{1/(n-1)}.
\]

Equation (15) allows us to reduce the set of five equations of motion to only three belonging to the big granules. When doing so, we obtain a new effective interaction between the large granules that indirectly accounts for the small granules that have been eliminated. However, this is not yet the complete transformation. While this step takes into account the structural effect of the small granules through an effective potential, in removing the small masses the mass of the entire system has been reduced. We need to take care of the “lost” mass, that is, we need to suitably renormalize the masses of the bigger granules in the new undecorated chain. This can be done by going back to the original five-granule chain and noting that the identity

\[
m_{k-2} \ddot{\bar{x}}_{k-2}(t) + m_{k-1} \ddot{\bar{x}}_{k-1}(t) + m_{k} \ddot{x}_k(t) + m_{k+1} \ddot{\bar{x}}_{k+1}(t) + m_{k+2} \ddot{\bar{x}}_{k+2}(t) = 0
\]

must hold at all times. This is the statement that the net force on the system is always zero. Using (14) we can again remove the variables corresponding to the small granules. The statement of zero force can then be expressed in terms of only three granules, a “left”, a “right” and a “middle” with masses \( \mu_l \), \( \mu_r \) and \( \mu_m \), respectively,

\[
\mu_m = m_k + \frac{m}{1 + 1/\alpha_{k-1}} + \frac{m}{1 + \alpha_{k+1}},
\]
Finally, a decorated chain of arbitrary length is then replaced by an effective undecorated chain with the following characteristics. The leftmost and rightmost masses are $\mu_l$ and $\mu_r$ respectively, each modified from the original end masses by a single small mass. The interior granules have masses $\mu_m$, modified from the original $m_k$ by the contributions of the two small masses that used to be on either side. The effective interaction mentioned above is most easily written if we relabel $(l, m, r)$ as new indices $(k-1, k, k+1)$ and is given by

$$V_{\text{eff}} = \zeta_k(n)(x_k - x_{k+1})^n, \quad \zeta_k(n) = \left[ \left( \frac{1}{r_k} \right)^{1/(n-1)} + \left( \frac{1}{r_{k+1}} \right)^{1/(n-1)} \right]^{1-n},$$

with $r_k = \sqrt{2rR_k/(r+R_k)}$.

Finally, we return to the frequency of the small “eliminated” granules. As noted earlier, when the small granule $(k-1)$ oscillates at maximum frequency, the force $F$ on the two sides cancel each other, i.e., $F = r_k(x_{k-2} - x_{k-1})^{n-1} = r_{k-1}(x_{k-1} - x_k)^{n-1}$. We can then estimate the maximum frequency of oscillation of a small granule in terms of the maximum force experienced by it. We find

$$\omega^2 = \frac{n-1}{m} \left[ \left( \frac{rR_{k-2}}{r+R_{k-2}} \right)^{1/(2n-2)} + \left( \frac{rR_k}{r+R_k} \right)^{1/(2n-2)} \right] \mathcal{F}^{n-2}_{n-1}.$$

Here the index $k$ is of course that of the original five-granule chain. In the following we will derive an expression for the maximum force $\mathcal{F}$ within the binary collision approximation.

The BCA results for the effective chain follow immediately from our earlier results (5) and (8) with the substitution of the new effective masses and interactions into the earlier equations. There is thus no need to explicitly rewrite these equations here. The only additional information here concerns the small granules – while they have been eliminated from the equations of motion, it is interesting to test how well the theory predicts the small granule oscillation frequency. When two granules collide, the maximum force corresponds to the maximum compression between them. Then from the equation of motion of two granules in the effective chain, it is straightforward to write $\mathcal{F} = \zeta_k(n)(z_k^{\text{max}})^{n-1}$. For the case of a decorated chain of just three granules (one small granule surrounded by two large ones), this result together with (18) gives the maximum frequency of oscillation of the small granule,

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\left(1 + \frac{2}{m}\right)(n-1) \left( \frac{2}{m+2} \sqrt{\frac{2r}{1+r}} \right)^{1/n} \left( \frac{n}{8} \frac{n-2}{2n^2} \right)}.$$

where $m$ and $r$ are the mass and radius of the small granule (in reduced units).
Comparison With Numerical Results

In this section we present highlights of three separate comparisons with exact numerical simulation results. Firstly, in a chain of only three granules (two large and one small) we analyze how well we capture the behavior of the small granule in the process of eliminating it from the equations of motion. Secondly, we compare numerical solutions for a long decorated chain with those of the associated effective chain in order to assess how well the effective chain represents the original decorated chain. Thirdly, we compare results of the BCA applied to the effective chain with exact numerical simulation results for the original chain. In all comparisons the large masses as well as the small masses are monodisperse.

We start with the decorated chain of only three granules. The solution of the exact dynamical equations (2) for this system is shown in the left panel of Fig. 5, where we report the variation in the displacement of the granules as a function of time. The monotonic middle curve is the average displacement of the small granule \( k \), which oscillates around this value throughout the process of collision. This figure conveys another important message; it justifies our neglecting the higher order terms in the Taylor expansion of the displacement of the small granule. The amplitude \( A \) here denotes the maximum distance between the average and maximum displacement of the small granule. This is clearly much smaller than the distance between the dashed curve and the displacement curves for the two large granules (except at the beginning and end of the collision process). Still continuing with the three-granule chain, we compare the solution (numerical and analytical are the same in this case) for the two-granule effective chain with the numerical solution for the three-granule decorated chain. Specifically, in the middle panel of Fig. 5 we display the granules’ velocity profiles as a function of time. The solid curves correspond to the decorated chain, and the dashed curves to the effective chain. The dynamics of the large granules are seen to be quite accurately described by the effective mapping.

Finally, we see that the small granule oscillates several times before the chain disintegrates. As a last comparison for the small decorated chain, we present the frequency of oscillation of the small granule as a function of its radius. The solid curve in the right panel of Fig. 5 represents the change in the average angular maximum oscillation frequency of the small granule predicted by the BCA in (19). The numerical data points are obtained by fitting the numerical solution of the equation of motion for the velocity profile of the small granule with the function \( v(t) = 0.5 + \omega A \cos(\omega t + \phi) \) (small granule oscillates around \( v = 0.5 \)). The results are in good qualitative agreement. While it may appear that the two results are in better agreement for larger values of \( r \), the percentage difference is 20% for the largest value of \( r \) as compared to 14% for the smallest \( r \).

Our second comparison is between the numerical integration results for a long decorated chain and the numerical integration results of the effective chain that replaces it. In the left panel of Fig. 6 we make this comparison for a small stretch of chain, but the agreement is typical of the entire chain. The results from the effective dynamics are in excellent agreement with the exact solution.

Finally, we present a comparison between the exact results for the time that it takes the pulse to reach large granule \( k \) as obtained from the exact equations of motion for the decorated chain with the result of the BCA. For the effective chain associated with the
FIGURE 5. Left panel: time dependence of the displacement of the three granules $k-1, k$, and $k+1$ (solid curves, from top to bottom) in a three-granule decorated chain. The monotonic middle curve denotes the average displacement $\bar{x}_k(t) = (x_{k-1}(t) + x_{k+1}(t)) / 2$ of the small granule. Middle panel: velocity profile of three granules obtained from the solution of the exact equations of motion (solid curves). The dashed curves represent results from the numerical solution of the equations of motion for the large granules in the effective chain. Note that the small granule of radius $r = 0.2$ oscillates with almost constant frequency while the amplitude shows a weak time dependence. Right panel: average angular maximum frequency of oscillation of the small granule obtained from the BCA using the effective interaction (solid curve), and the numerical solution of the exact equations of motion of the three-granule decorated chain (dots). Figures from Ref. [7].

original decorated chain, the time $T_k$ spent on relabeled granule $k$ is independent of $k$ (except for the edge granules, which have a different effective mass). It is given by

$$T_k = \sqrt{\pi} \left( \frac{n^{2n-3} \mu}{\mathcal{R}^{n-1}} \right)^{1/n} \frac{\Gamma(1 + 1/n)}{\Gamma(1/n + 1/2)}.$$  \hspace{1cm} (20)

Here $\mu$ is the effective mass $\mu = 1 + m$ and $\mathcal{R} = [2r/(1 + r)]^{1/2}$, where $m$ is the mass and $r$ the radius of the small granules in the original chain. The time $t = \sum_{k'=1}^{k} T_{k'}$ therefore varies linearly with $k$. The slope of the line is determined by $T_k$. In the right panel of Fig. 6 we show numerical results for the time taken by the pulse to reach the $k$th granule. For clarity we show results for the restricted range $20 < k < 50$. Various symbols represent different values of $r$. In the same figure we also show the analytic result of the binary collision approximation. No fitting parameters are involved, and the agreement of the two results is again gratifying. This shows that the BCA can give quantitatively reliable results for simple decorated chains. However, there is one limitation that we have respected here: the radius of the small granules must be no larger than $\sim 40\%$ of that of the larger granules [7, 8].

CONCLUSIONS

We have studied pulse propagation in a variety of one-dimensional granular chains. Our goal has been to develop an analytic approximation that makes it possible to predict pulse propagation behavior and to compare the results of our analytic procedure with exact results obtained from numerical simulations and from experiments. Our work is based on a binary collision assumption whereby the pulse travels down the chain via a succession of two-granule collisions. We showed that this approximation is quantitatively excellent for linearly forward and backward tapered chains. We note that equal agreement is achieved for geometrically tapered chains (and, of course, for the even smoother case of monodisperse chains). We also showed that a generalization of the
FIGURE 6. Left panel: exact results (solid curves) for the velocity profiles of the larger granules in a simple decorated chain for $r = 0.3$. The dotted curves show results from the solution of the effective dynamical equations. Right panel: time $t$ taken by the pulse to reach the $k$th granule in a simple decorated chain for $r = 0.1$ (open circles), $r = 0.2$ (filled circles), $r = 0.3$ (squares), and $r = 0.4$ (plus signs). The theoretical results are shown with dashed ($r = 0.1$), dashed-dotted ($r = 0.2$), dotted ($r = 0.3$), and continuous ($r = 0.4$) lines. Figures from Ref. [7].

approximation to decorated chains yields equally quantitatively accurate results. We have also applied this methodology to randomly decorated chains, that is, to chains with small granules of randomly varying radii [8]. Furthermore, we have studied granular chains with momentum-conserving and also with momentum non-conserving dissipation [22, 26, 27], precompressed chains, chains subject to gravitational forces [21], and chains of o-rings in which the potential is a sum of terms with different powers [21]. We have also studied heat propagation in one-dimensional granular gases [1, 25, 20]. One of our ultimate goals is to be able analytically optimize the granular distribution along the chain for a particular desired purpose [5].

Beyond this, we have begun to extend our studies to two-dimensional granular beds. Here the problem is immediately made much more difficult by the fact that different packings are possible, and that translational and rotational energy can be interconverted in granular collisions in this case. These added difficulties pose interesting challenges yet to be addressed in any analytical work.

ACKNOWLEDGMENTS

Acknowledgment is made to the Donors of the American Chemical Society Petroleum Research Fund for partial support of this research (K.L. and U.H.). A.R. acknowledges support from Bionanotec-CAPES and CNPq. A.H.R. acknowledges support by CONACyT Mexico under Projects J-59853-F and J-83247-F.

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