Effects of disorder on synchronization of discrete phase-coupled oscillators

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We study synchronization in populations of phase-coupled stochastic three-state oscillators characterized by a distribution of transition rates. We present results on an exactly solvable dimer as well as a systematic expansion of our earlier studies to the arena of transition rate disorder. We also provide results for globally coupled arrays where the transition rate of each unit is drawn from a uniform distribution of finite width. Even in the presence of transition rate disorder, numerical and analytical results point to a single phase transition to macroscopic synchrony at a critical value of the coupling strength. Numerical simulations make possible further characterization of the synchronized arrays.

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I. INTRODUCTION

Synchronization in populations of phase-coupled nonlinear stochastic oscillators and the corresponding emergence of macroscopic coherence appear pervasively in a tremendous range of physical, chemical, and biological systems. As a result, the general subject continues to be studied intensely in applied mathematics and theoretical physics [1–5]. Since the pioneering work of Kuramoto [3], emergent cooperation in these systems has been investigated from a myriad of perspectives encompassing both globally and locally coupled, stochastic and deterministic, and large and small systems. And while Kuramoto’s canonical model of nonlinear oscillators, whose use is widespread because of its close kinship to the normal form describing general phase oscillations, has proven spectacularly successful for characterizing synchronization, simple, phenomenological models of synchronization have also proven useful in a variety of new contexts [6–9], most notably the characterization of emergent synchronization as a nonequilibrium phase transition.

We have shown [8,9] that a model of three-state identical phase-coupled stochastic oscillators is ideally suited for studying the nonequilibrium phase transition to synchrony in locally coupled systems, owing in large part to its numerical simplicity. The utility of these studies rests on the well-established notion of universality—that is, on the contention that microscopic details do not determine the universal properties associated with the breaking of time-translational symmetry that leads to a macroscopic phase transition. Statistical mechanics is thus enriched by simplistic, phenomenological models (the Ising model being the most ubiquitous example) whose microscopic specifics are known to be, at best, substantial simplifications of the underlying quantum mechanical nature of matter, but whose critical behavior captures that of more complex real systems. In this spirit, our simple tractable model captures the principal features of the synchronization of phase-coupled oscillators. In the globally coupled (mean-field) case our model undergoes a supercritical Hopf bifurcation. With nearest-neighbor coupling, we have shown that the array undergoes a continuous phase transition to macroscopic synchronization marked by signatures of the XY universality class [10,11], including the appropriate classical exponents $\beta$ and $\nu$ and lower and upper critical dimensions 2 and 4, respectively.

In this paper we focus on globally coupled arrays and expand our earlier studies to the arena of transition rate disorder. We start with a modified version of our original model (explained below), in which identical synchronized units are governed by the same transition rates as individual uncoupled units. Then, in the spirit of the original Kuramoto problem [3], we explore the occurrence of synchronization when there is more than one transition rate and perhaps even a distribution of transition rates among the phase-coupled oscillators. In particular, we explore the conditions (if any) that lead to a synchronization transition in the face of a transition rate distribution, discuss the relation between the frequency of oscillation of the synchronized array and the transition rates of individual units, and explore whether or not the existence of units of different transition rates in the coupled array may lead to more than one phase transition.

Our paper is organized as follows. In Sec. II we describe the model, including the modification (and associated rationale) of our earlier scenario even for an array of identical units. In Sec. III we present results for a dimer composed of two units of different intrinsic transition rates. While there is of course no phase transition in this system, it is instructive to note that there is a probability of synchronization of the two units that increases with increasing coupling strength. Section IV introduces disorder of a particular kind, useful for a number of reasons that include some analytical tractability. Here our oscillators can have only one of $\mathcal{N}$ distinct transition rates, where $\mathcal{N}$ is a small number. We pay particular attention to the dichotomous case $\mathcal{N}=2$. This simple disordered system reveals some important general signatures of synchronization. We also consider the cases $\mathcal{N}=3$ and $\mathcal{N}=4$, but find that the $\mathcal{N}=2$ case already exhibits most of the interesting qualitative consequences of a distribution of tran-
coupling in square, cubic, or hypercubic arrays, \( n = 2d \) (\( d \) = dimensionality). Here we focus on the globally coupled array.

For a population of \( N \rightarrow \infty \) identical units in the mean-field (globally coupled) version of this model we can replace \( N_i/N \) with the probability \( P_i \), thereby arriving at a nonlinear equation for the mean-field probability, \( \partial P(t)/\partial t = M[P(t)]P(t) \), with

\[
M[P(t)] = \begin{pmatrix}
-g_{12} & 0 & g_{31} \\
g_{12} & -g_{23} & 0 \\
0 & g_{23} & -g_{31}
\end{pmatrix}.
\]

Normalization allows us to eliminate \( P_j(t) \) and obtain a closed set of equations for \( P_1(t) \) and \( P_2(t) \). We can then linearize about the fixed point \((P_1^*, P_2^*) = (1/3, 1/3)\), yielding a set of complex conjugate eigenvalues which determine the stability of this disordered state. Specifically, we find that \( 2\lambda_2/g = (a-3) \pm i\sqrt{3(1+a)} \), eigenvalues that cross the imaginary axis at \( a_c = 3 \), indicative of a Hopf bifurcation at this value. Note that the oscillation frequency of the array at the critical point as given by the imaginary parts of the eigenvalues is \( \omega = 2\sqrt{3}g \).

Our previous work used a different form of the coupling (2), with \( N_i \rightarrow N_i \). That earlier choice was numerically advantageous because it led to a phase transition at a lower critical value \( a_c \) of the coupling constant \( (a_c = 1.5 \text{ in the globally coupled array}) \) than other choices. A lower coupling in turn facilitates numerical integration of equations of motion because the time step that one needs to use near the phase transition must be sufficiently small, \( dt \ll e^{-a/1}g \). However, that earlier coupling choice brought with it a result that is undesirable in our present context (but was of no consequence before). In our earlier model, as the units become increasingly synchronized above the transition point, the average transition rate of a cluster becomes substantially dependent on the value of \( a \); specifically, the transitions and cluster oscillation frequency slow as \( a \) is increased due to an exponential decrease in the transition probability. To cite an explicit example, consider a small subsystem composed of units which are all in the same state at time \( t \) (that is, a cluster of units which are perfectly synchronized). The previous form of the coupling yields an exponentially small transition rate in this case, and hence the oscillation frequency of this microscopic cluster approaches zero for high values of \( a \). Since here we specifically wish to analyze the effects of transition rate disorder, it is desirable to deal with a model in which the average transition rate of identical synchronized units depends only on their intrinsic transition rate parameter and not on coupling strength. The form (2) reduces simply to the constant \( g \) when the coupled units are perfectly synchronized.

III. DIMER

Consider first the simplest “disordered array;” namely, a mutually coupled dimer where one unit is characterized by \( g_1 = g_1 \) and the other by \( g_2 = g_2 \). In terms of the states (phases) \( S_1 \) and \( S_2 \) of units 1 and 2, there are nine possible dimer
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states, \((S_1, S_2) = (1, 1), (1, 2), \ldots, (3, 3)\), but it is not necessary in order to seek the ensemble distributions for all of these states to decide whether or not the two units are synchronized. We can directly write an exact reduced linear evolution equation for the three states \(A, B,\) and \(C\), where \(A\) corresponds to any situation where both units are in the same state [that is, \((S_1, S_2) = (1, 1), (2, 2),\) and \((3, 3)\)], state \(B\) corresponds to a situation where unit 1 is one state “ahead” \([(S_1, S_2) = (2, 1), (3, 2)\) and \((1, 3)\)], and state \(C\) corresponds to a situation where unit 2 is one state “ahead” \([(S_1, S_2) = (1, 2), (2, 3),\) and \((3, 1)\)]. The evolution equation for these states is the closed linear set

\[ \frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{A} \mathbf{P}(t), \]  

with \(\mathbf{P}(t)\) the time-dependent probability column vector \((\mathbf{P}_A(t) \mathbf{P}_B(t) \mathbf{P}_C(t))^T\) and

\[ \mathbf{A} = \begin{pmatrix} -\gamma_1 - \gamma_2 & b \gamma_2 & b \gamma_1 \\ \gamma_1 & -b^{-1} \gamma_1 - b \gamma_2 & b^{-1} \gamma_2 \\ \gamma_2 & b^{-1} \gamma_1 & -b \gamma_1 - b^{-1} \gamma_2 \end{pmatrix}, \]  

and where we have introduced the abbreviation

\[ b = e^a. \]  

This evolution equation is easy to derive from the definition of the coupling, Eq. (2). For example, when the system is in state \(A\), it can either go to state \(B\), which happens when unit 1 jumps ahead with transition rate \(\gamma_1\), or it can go to state \(C\), which happens when unit 2 jumps ahead with transition rate \(\gamma_2\). Similarly, when the system is in state \(B\), it can either jump to state \(A\) (when the lagging unit transitions forward) with transition rate \(b \gamma_2\) or jump to state \(C\) (when the leading unit transitions forward) with transition rate \(b^{-1} \gamma_1\).

With normalization, Eq. (4) becomes a two-dimensional equation having steady-state solution

\[ \mathbf{P}^* = \frac{b \gamma_1 \gamma_2 + \gamma_2^2}{(1 + b + b^2) \gamma_2 + (2 + b^2) \gamma_1 \gamma_2}, \]

\[ \mathbf{P}^*_B = \frac{b \gamma_1 \gamma_2 + \gamma_2^2}{(1 + b + b^2) \gamma_2 + (2 + b^2) \gamma_1 \gamma_2}. \]  

The eigenvalues of the two-dimensional matrix obtained from \(\mathbf{A}\) after implementing normalization have negative real parts for all positive values of the parameters \(a, \gamma_1,\) and \(\gamma_2\), indicating that the fixed points given by Eq. (7) are stable. Hence, the system asymptotically tends to this steady-state solution. We are particularly interested in \(\mathbf{P}^*_A\), the probability for the system to be synchronized. In terms of the single relative width parameter

\[ \mu = \frac{2 |\gamma_1 - \gamma_2|}{(\gamma_1 + \gamma_2)} \]  

\((0 \leq \mu \leq 2)\), this probability is

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function of \( a \) in the infinite systems to be considered below. The decreased synchronization probability when the frequencies of the two units become more dissimilar (increasing \( \mu \)) will also be reflected in the dependence of the critical coupling on transition rate parameter disorder.

**IV. \( N \) DIFFERENT TRANSITION RATES**

Next we consider globally coupled arrays of oscillators that can have one of \( N \) different transition rate parameters, \( g = \gamma_u, \ u = 1, \ldots, N \). To arrive at a closed set of mean-field equations for the probabilities we again go to the limit of an infinite number of oscillators, \( N \to \infty \). However, we must do so while preserving a finite density of each of the \( N \) types of oscillators. The probability vector is now \( 3N \) dimensional, \( P(t) = (P_1, \gamma_1 P_2, \gamma_1 P_3, \gamma_1 \ldots P_1, \gamma_1 P_2, \gamma_1 P_3, \gamma_1)^T \). The added subscript on the components of \( P(t) \) keeps track of the transition rate parameter. Explicitly, the component \( P_i, \gamma_u \) is the probability that a unit with transition rate parameter \( \gamma_u \) is in state \( i \). The mean-field evolution for the probability vector is the set of coupled nonlinear differential equations \( dP(t)/dt = M[M[P(t)]P(t) \), with

\[
M[M[P(t)]P(t)] = \begin{pmatrix} M_{\gamma_1} & 0 & \cdots & 0 \\ 0 & M_{\gamma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{\gamma_N} \end{pmatrix}. \tag{10}
\]

Here

\[
M_{\gamma_u} = \begin{pmatrix} -g_{12}(\gamma_u) & 0 & g_{31}(\gamma_u) \\ g_{12}(\gamma_u) & -g_{23}(\gamma_u) & 0 \\ 0 & g_{23}(\gamma_u) & -g_{31}(\gamma_u) \end{pmatrix} \tag{11}
\]

and

\[
g_{ij}(\gamma_u) = \gamma_u \exp \left[ a \sum_{k=1}^{N} \phi(\gamma_k)(P_{i,\gamma_k} - P_{i-1,\gamma_k}) \right] \delta_{j,i+1}. \tag{12}
\]

The function \( \phi(\gamma_k) \) is the fraction of units which have a transition rate parameter \( g = \gamma_u \).

We focus on uniform distributions \( \phi(\gamma_u) = 1/N \), but subsequently take note of the consequences of relaxing this assumption. For uniform distributions, probability normalization again allows us to reduce this to a system of 2\( N \) coupled ordinary differential equations. It is interesting to compare this setup with that of the original Kuramoto problem with noise, where a continuous frequency distribution is introduced and the governing equation is a nonlinear partial differential equation [the Fokker-Planck equation for the density \( \rho(\theta, \omega, t) \)] [12]. The discretization of phase in our model results instead in a set of 2\( N \) coupled nonlinear ordinary differential equations.

While it is nevertheless still difficult to solve these equations even for small \( N \), we can linearize about the disordered state \( P(t) = (1/3, 1/3 \ldots 1/3)^T \) and arrive at a 2\( N \times 2N \) Jacobian of the block matrix form

\[
J = \begin{pmatrix} J_1(\gamma_1) & J_1(\gamma_1) & J_1(\gamma_1) & \cdots & J_1(\gamma_1) \\ J_2(\gamma_2) & J_2(\gamma_2) & J_2(\gamma_2) & \cdots & J_2(\gamma_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_2(\gamma_N) & J_2(\gamma_N) & \cdots & J_2(\gamma_N) & J_1(\gamma_N) \end{pmatrix}. \tag{13}
\]

The blocks \( J_1(g) \) and \( J_2(g) \) are given by

\[
J_1(g) = \begin{pmatrix} -2g & -g - ag/N \\ g + ag/N & -g + ag/N \end{pmatrix} \tag{14}
\]

and

\[
J_2(g) = \begin{pmatrix} 0 & -ag/N \\ ag/N & ag/N \end{pmatrix}. \tag{15}
\]

While we explore this in more detail below only for small \( N \), we note that in general the Jacobian (13) has \( N \) pairs of complex conjugate eigenvalues, only one pair of which seems to have a real part that becomes positive with increasing coupling constant \( a \). This implies that there is a single transition to synchrony even in the presence of the transition rate disorder that we have introduced here. We go on to confirm this behavior for \( N = 2, 3, \) and \( 4 \).

**A. Two transition rate parameters**

For the \( N = 2 \) case, the four eigenvalues \( (\lambda_+, \lambda_+, \ldots, \lambda_\ldots) \) of the Jacobian can be determined analytically. We find

\[
\frac{\text{Re} \lambda_+}{\gamma_1 + \gamma_2} = \frac{1}{8} \left( a - 6 \pm B(a, \mu) \cos[C(a, \mu)] \right),
\]

\[
\frac{\text{Im} \lambda_+}{\gamma_1 + \gamma_2} = \frac{1}{8} \left( \sqrt{3}(a + 2) \pm B(a, \mu) \sin[C(a, \mu)] \right), \tag{16}
\]

where
the synchronous and disordered phases. This contour represents the critical surface and thus separates the Hopf bifurcation occurs and the disordered solution becomes unstable. The region above the contour represents the synchronized phase. Lower panel: stability boundary in terms of relative width parameter.

FIG. 5. (Color online) Upper panel: stability boundary for the dichotomously disordered system. The contour $\text{Re} \lambda_*=0$ is plotted in $(\gamma_1, \gamma_2, a)$ space. This contour indicates the critical point where the Hopf bifurcation occurs and the disordered solution becomes unstable. The region above the contour represents the synchronized phase. Lower panel: stability boundary in terms of relative width parameter.

$$B(a, \mu) = \sqrt{2} [a^4 - 6a^2 \mu^2 + 3 \mu^4 (a^2 + 3)]^{1/4},$$

$$C(a, \mu) = \frac{1}{2} \tan^{-1} \left( \frac{-\sqrt{3} a^2 - (a + 3) \mu^2}{a^2 + 3(a - 1) \mu^2} \right).$$

Aside from an overall factor $(\gamma_1 + \gamma_2)$, Eqs. (16) depend only on the relative width variable as defined in Eq. (8), and therefore the critical coupling $a_c$ depends only on $\mu$. As illustrated in the upper panel of Fig. 14, below, one pair of eigenvalues acquires a positive real part (i.e., crosses the imaginary axis) at a critical value $a=a_c$, but the other pair shows no qualitative change as $a$ is varied. While this figure shows only the particular transition rate parameter values $(\gamma_1, \gamma_2)=(1, 3)$, the qualitative features of these eigenvalues remain similar for the entire range of positive parameters. The upper panel of Fig. 5 depicts the contour $\text{Re} \lambda_*=0$ in $(\gamma_1, \gamma_2, a)$ space; this contour represents the critical surface and thus separates the synchronous and disordered phases.

The critical coupling is the value of $a$ at which $\text{Re} \lambda_*=0$ (Re $\lambda_*$ does not vanish for any $a$). It is easy to ascertain that Im $\lambda_*$ does not vanish at $a_c$, so that the critical point is a Hopf bifurcation. Furthermore, it is clear from Eq. (16) that $a_c$ depends only on the relative width parameter $\mu$, and it is also straightforward to establish that $a_c$ increases with increasing $\mu$; that is, a stronger coupling is necessary to overcome increasingly different values of $\gamma_1$ and $\gamma_2$ (see lower panel of Fig. 5). Note, however, that the dependence on $\mu$ implies that it is not just the difference in transition rates but the relative difference or percent difference relative to the mean transition rate that is the determining factor in how strong the coupling must be for synchronization to occur. A small-$\mu$ expansion leads to an estimate of $a_c$ to $O(\mu^2)$,

$$a_c \approx \frac{1}{8} [12 + 3 \mu^2 + \sqrt{3} (12 + \mu^2)(4 + 3 \mu^2)],$$

a result that exhibits these trends explicitly. The upper panel in Fig. 6 shows that this estimate is remarkably helpful even when $\mu^2$ is not so small.

The frequency of oscillation of the synchronized system at the transition is given by $\omega=\lim_{a \rightarrow a_c} \text{Im} \lambda_*$. From Eq. (16) it follows that $\omega$ depends on $(\gamma_1 + \gamma_2)$ as well as $\mu$. The small-$\mu$ expansion leads to the estimate

$$\omega = \frac{1}{4} \sqrt{3} (\gamma_1 + \gamma_2)(4 + \mu^2),$$

which works exceedingly well up to $\mu \sim 1$ (see Fig. 6).
To check the predictions of our linearization procedure, we numerically solve the nonlinear $N=2$ mean-field equations. In agreement with the structure of the linearized eigenvalues, all components of $P(t)$ synchronize to a common frequency as the phase boundary in $(\mu, a)$ space is crossed. Interestingly, the numerical solutions also give us insight into the amplitude of the oscillations; that is, they allow us to explore the relative “magnitude” of synchronization within the two populations. As we will see, the two populations indeed oscillate with the same frequency, but with amplitudes and “degrees of synchronization” that can be markedly different. Consider the order parameter $r(t)$ given by

$$r(t) = \frac{1}{N} \left| \sum_{i=1}^{N} e^{i\phi_i} \right|.$$  \hspace{1cm} (20)

Here $\phi_i$ is the discrete phase $2\pi(k-1)/3$ for state $k \in \{1, 2, 3\}$ at site $\nu$. For phase transition studies, one would likely average this quantity over time in the long-time limit and also over independent trials. For our purposes here, though, we find the time-dependent form more convenient. In the mean-field case, where we solve for probabilities to be in each state, the order parameter is easily calculated by writing the average in Eq. (20) in terms of these probabilities rather than as a sum over sites.

In subsequent figure captions we introduce the notation $\gamma = (\gamma_1 + \gamma_2)/2$ (average transition rate parameter) and the difference $\Delta = |\gamma_2 - \gamma_1|$ (note that $\mu = \Delta/\gamma$). As shown in Figs. 7–11, the predictions of linearization accurately describe the onset of macroscopic synchronization and provide an estimation of the frequency of these oscillations near threshold (see Fig. 9). Specifically, Figs. 7 and 8 show the macroscopic oscillations for coupling $a$ above threshold. All the oscillators, regardless of their intrinsic transition rate parameter, oscillate exactly in phase, but the degree of synchronization is greater in the population with the larger $\gamma_i$ (here $\gamma_2$), as evidenced by the unequal amplitude of the components of $P(t)$ for the two populations. The “greater degree of synchronization” is also apparent in the order parameter $r(t)$.
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shown in the insets, which is larger for the oscillators with the higher intrinsic transition rate. These results support the notion that populations with higher transition rate parameters in some sense synchronize more readily. The figures also show the frequency spectrum of any component of $P(t)$. The peak occurs at the frequency of oscillation of the synchronized array. As $\alpha \rightarrow \alpha_c$, this frequency approaches the value $\text{Im} \lambda_+$ predicted by linearization, as shown in Fig. 9.

FIG. 10. (Color online) Time-averaged order parameter $r$ in the long-time limit vs $\Delta$ for the individual oscillator populations characterized, respectively, by the transition rate parameter $\gamma_1$ (stars) and $\gamma_2$ (circles) and for the entire mixed array (squares). The insets show the time evolution of the probability vector components $P_{1,\gamma_1}$ and $P_{2,\gamma_1}$ (lighter or brown curves) and $P_{1,\gamma_2}$ and $P_{2,\gamma_2}$ (darker or blue curves) for widths 0.05 (upper inset), 0.5 (middle inset), and 0.9 (lower inset). Some of the curves are not visible because they are so perfectly superimposed. While the degree of synchronization varies within each population, the critical width for desynchronization is the same for both, as predicted by linearization. The coupling constant for all cases is $\alpha=3.2$ and the average transition rate parameter $\gamma=1.5$.

FIG. 11. (Color online) Same as Fig. 10 but with $\gamma=3.5$ and widths 0.1 (upper inset), 1.3 (middle), and 1.8 (lower).

FIG. 12. (Color online) Frequency spectra of the numerical solutions to the mean-field equations ($N=2$) show that above critical coupling, $\alpha=3.2>\alpha_c=3.06$, synchronization occurs at a frequency closer to the lower of the two population frequencies. Top left inset: $P_{1,\gamma=3.5}$ and $P_{1,\gamma=4.5}$ when all units are globally coupled. Bottom right inset: the same curves for populations that are uncoupled from one another (but still globally coupled within each population).

Figures 10 and 11 illustrate the sudden desynchronization (at fixed $\alpha$ and $\gamma$) accompanying an increase in the difference $\Delta$. This behavior is reminiscent of that of the original Kuramoto oscillators, which become disordered as the width of the frequency distribution characterizing the population exceeds some critical value. The insets show the components of $P(t)$ and confirm that both populations undergo the desynchronization transition at the same critical value of the difference $\Delta$. Comparing the two figures, we see that the system with a higher average transition rate parameter (Fig. 11) can withstand a larger difference $\Delta$ before desynchronization, again confirming our earlier observations.

One last point to consider is the relation between the frequency of oscillation of the synchronized array above $\alpha_c$ and the frequencies of oscillation of the two populations if they were decoupled from one another. As coupling increases, the oscillation frequency $\omega$ moves closer to that of the population with the lower transition rate parameter. This is illustrated in Fig. 12 for the same parameters used in Fig. 9.

Finally, a visually helpful illustration of these behaviors is obtained via a direct simulation of an array with a dichotomous population of oscillators. Since our oscillators are globally (all-to-all) coupled, the notion of a spatial distribution is moot, and for visualization purposes we are free to arrange the populations in any way we wish. In Fig. 13 we display an equal number of $\gamma_1$ and $\gamma_2$ oscillators and arrange the total population of $N=5000$ so that the first 2500 have transition rate parameter $\gamma_1$ and the remaining 2500 have transition rate parameter $\gamma_2$. In this simulation we have chosen $\gamma_1=0.5$ and $\gamma_2=1.5$, so that $\gamma=\Delta=1$. Although $N=5000$ is not infinite, it is large enough for this array to behave as predicted by our mean-field theory. The left panel shows snapshots of the phases (each phase is indicated by a different color online) when $\alpha<\alpha_c$ and the phases are ran-
In both cases, all units are globally coupled. For visualization purposes, the plot is arranged so that population $\gamma_1$ consists of the first 2500 units (the top), while population $\gamma_2$ consists of the second 2500 units (the bottom). Global synchrony emerges for $a > a_c$. In addition, the population with the higher-transition-rate parameter is more synchronized.

dom. The right panel shows the synchronized array when $a > a_c$. Clearly, all units are synchronized in the right panel, but the population with the higher-transition-rate parameter (lower half) shows a higher degree of synchronization [higher $P(t)$] as indicated by the intensity of the colors or the gray scale.

B. $N=3$ and $N=4$

We can carry out this analysis, albeit not analytically (at least in practice), for any $N$. We have explored the cases $N=3$ and $4$. In both cases there appears to be only one pair of eigenvalues whose real parts can become positive, suggesting that synchronization occurs all at once and not in one population at a time (middle and lower panels of Fig. 14). This occurs no matter the distribution of the three or four transition rate parameters. For example, in the $N=4$ case we have compared in some detail the cases where the four transition rates are equidistant and where they are pairwise much closer than the separation between the highest and lowest. In both cases there is a single transition to synchrony, albeit not at exactly the same value of $a_c$, indicating a more complex dependence on the transition rate parameter distribution than just via its mean and width. Furthermore, the basic trends of the dichotomous case broadly carry over, mainly in that the critical value $a_c$ increases when the width of the distribution increases relative to the mean (as one would expect). On the other hand, the inclusion of more transition rates within a given range leads to a lowering of the critical coupling. Thus, for example, the mean transition rate $\gamma$ and the width $\Delta$ are the same in the cases shown in the upper middle panels of Fig. 14 ($\gamma=\Delta=2$), and yet $a_c$ is higher in the former (3.95 for $N=2$) than in the latter (3.6 for $N=3$). Still, the mean and width of the distribution provide a rough qualitative assessment of the behavior, particularly for the case of a continuous uniform distribution, which we study in the next section.

In this analysis we have focused on disorder with a uniform distribution; that is, each of the $N$ frequency parameters is represented by a fraction $1/N$ of the population of oscillators. How robust are our results to a change in this distribution? In particular, if the frequency parameters are not equally represented, will there still be a single transition to synchronization? Will the mean and width of the distribution still be the principal measures of the qualitative behavior of the system? While we leave most of this analysis to future work, we are willing to make a conjecture about the generalization of the principal finding of this work: that there will...
still be a single transition to synchronization regardless of the distribution. In fact, we conjecture that the strongest “conflict” for the system is posed by a uniform distribution, since any other will have majority populations that are more likely to be more determinant of the behavior of the system. To support this conjecture, we present in Figs. 15 and 16 illustrative results for nonuniform dichotomous populations. In Fig. 15 we show the stability boundary for varying population ratios as a function of the coupling parameter. In Fig. 16 we show the real parts of each of the two pairs of eigenvalues with increasing coupling for a number of population ratios, as detailed in the caption. Again, only the real parts of one pair of eigenvalues cross from negative to positive, indicating a single transition to synchronization for the entire system. Note that the critical coupling approaches ac = 3 at each population extreme (consistent with the fact that ac in an array of identical oscillators is independent of the frequency parameter [8,9]) and that, as conjectured, the strongest coupling is required for synchronization when the two populations are equally represented. While Figs. 15 and 16 only show results for (γ1, γ2) = (1, 2), we obtain similar results for other pairs of frequency parameters within the range of values used in other figures in this paper.

FIG. 15. Stability boundary in terms of population ratios φ(γ1)/φ(γ2) of oscillators with frequency parameters γ1 = 1 and γ2 = 2.

FIG. 16. Real parts of the complex eigenvalues for two populations of oscillators with frequency parameters γ1 = 1 and γ2 = 2. The population ratios φ(γ1)/φ(γ2) are 99/1, 3/1, 1/1 (the case analyzed in detail in the paper), 1/3, and 1/99.

V. UNIFORM DISTRIBUTION OF TRANSITION RATE PARAMETERS

We now turn to globally coupled arrays where the transition rate parameter g for each unit is chosen from a uniform distribution over a finite interval, φ(g). While it is difficult to make direct analytical progress in this general case, the earlier dimer analysis and the arrays of N = 2, 3, 4 different populations of units provide a framework for understanding the properties of these more general systems. In particular, the earlier results suggest that this “more disordered” system may also display a single transition to synchronization. To explore these and other features in more detail, we simulate N = 5000 globally connected units characterized by the transition rate parameter distribution φ(g) and we make several observations. First, we observe a single transition to macroscopic synchronization. Second, as suggested by the dichotomous case, synchronization appears more readily (that is, for a lower value of a) if the distribution φ(g) has a larger mean and smaller width. When the mean and width are varied independently, the qualitative trends from the dichotomous case are observed here as well. Third, while synchronization in this system is again governed primarily by the mean and width of the distribution φ(g), the critical value ac is considerably lower than that of the finite N systems with the same mean and width (as expected).

Two examples of our simulation results are shown in Figs. 17 and 18. In Fig. 17 we present the first two components of the three-dimensional vector P(t) whose components P1(t) and P2(t) represent the probability that all units of the entire synchronized array are in state i. The probabilities P1(t) and P2(t) oscillate in time with slightly modulated amplitudes and a constant relative phase, indicating global synchronization. The upper left inset shows the order parameter r(t) and the upper right inset the time-resolved snapshot of the system.
that the dichotomous array as well as the dimer synchroni-
critical value
distributions with different means. As the mean of the
curves represent the steady-state, time-averaged order parameter for
is destroyed. The coupling is chosen to be
increases, a critical width is reached beyond which synchronization
population of mean transition rate parameter 3.5
bution increases, the transition to disorder occurs at a greater width.
the triangle order parameter data
FIG. 19. (Color online) As the width of the \( \phi(g) \) distribution
both indicating a high degree of synchronization. Note that
the coupling parameter \( a = 3.15 \) in the figure is below the
critical value \( a_c = 3.2 \) for the dichotomous case with the same
mean and width.

Figure 18 shows the steady-state time-averaged order
parameter \( r \) at constant \( a \) as the width of the \( \phi(g) \) distribution
increased for a fixed mean. Similar to the \( N = 2 \) population
case, synchronization is destroyed as the width eclipses some
critical value and that value increases as the mean of the
distribution increases. In Fig. 19 we plot the data from Fig.
18 as a function of the relative width parameter \( \mu \). Recalling
that the dichotomous array as well as the dimer synchroni-
ization at a given \( a \) depends only on \( \mu \), we might expect that
the transition point \( \mu_c \) (at constant \( a \)) is not significantly
mean dependent, even when there is a distribution of transition
rate parameters. In fact, we can see that the curves approximate collapse onto one curve, indicating that the relative
width \( \mu \) provides a useful control parameter for predicting synchronization. Hence, the predictions of the line-
earization analysis for the \( N = 2 \) case provide qualitative insight into the behavior of the disordered population.

VI. DISCUSSION

We have presented a discrete model for globally coupled
stochastic nonlinear oscillators with a distribution of transition
rate parameters. Our model exhibits a range of interesting
dynamical behavior, much of which mimics the qualitative features of the canonical Kuramoto oscillator [3], but
with a mathematically and numerically considerably more
tractable model. Since our phase variable is discrete
(wheras the phase variable in the canonical problem is continuous), a distribution of \( N \) different transition rates in our
array leads to a set of \( 2N \) coupled nonlinear ordinary differential equations instead of a single partial differential equation for the probability distributions of interest. Linearization of our model around the critical point leads to a problem which at least for small \( N \) (specifically, for the dichotomous disorder case) becomes analytically tractable. Distributions involving a large finite number of transition rate parameters, while not easily amenable to analytic manipulation even
upon linearization, reduce to a simple matrix algebra problem.
For any distribution of transition rate parameters, even
continuous, the model is in any case readily amenable to
numerical simulation.

Our most salient conclusion is that such disordered glo-
bally coupled arrays of oscillators, even in the face of transition
rate parameter disorder, undergo a single transition to
macroscopic synchronization. While the coupling is entirely
different in the canonical Kuramoto model and therefore
direct comparisons with our model are not straightforward, we
note that a single transition involving a single discrete eigen-
value emerging from a continuous spectrum is observed for a
continuum of frequencies distributed unimodally and asym-
metrically [3,12–14]. We have also shown that the critical
coupling \( a_c \) for synchronization in our model depends
strongly (but not exclusively) on the width \( \Delta \) and mean \( \gamma \)
of the transition rate parameter distribution, specifically via the
relative width \( \mu = \Delta / \gamma \). This general feature is already appar-
ent in the synchronization behavior of a dimer of two oscil-
lators with transition rate parameters \( \gamma_1 \) and \( \gamma_2 \). An infinite
array of two populations of oscillators, one with transition
rate parameter \( \gamma_1 \) and the other with \( \gamma_2 \), displays a Hopf
bifurcation, with \( a_c \) determined solely by \( \mu \). While a quanti-
tative prediction of synchronization on the basis of the relative
width is not possible in all cases, it does determine qualitative aspects of the transition for more complex transition
rate parameter distributions. We have explored this assertion for arrays with \( N = 2, 3, \) and 4 and with a uniform
distribution of transition rates over a finite interval, and ex-
pect it be appropriate for other smooth distributions as well.
In the Kuramoto model the distribution of frequencies is usually taken to be Lorentzian for analytic tractability and therefore determined by a single parameter that precludes separate characterization in terms of two parameters (average frequency and width).

A number of further avenues of investigation based on our stochastic three-state phase-coupled oscillator model are in progress. Most interestingly, the Hopf bifurcation characterizing the transition to synchrony in our modified model appears subcritical when the coupling parameter is very close to the critical coupling. This suggests that the emergence of synchronization in this regime may be marked by characteristics of a first-order phase transition, including multistability. Characterization of this new regime for arrays of identical oscillators as well as disordered arrays is in progress. In addition, we are currently exploring the effects of transition rate disorder in locally coupled arrays. In this case, even the dichotomously disordered array is further complicated by the fact that the spatial distribution of the two populations may play a role. Analysis of globally coupled arrays with more complex distributions of frequency parameters is underway to follow up the preliminary results presented here for dichotomous systems. The study of locally coupled arrays with more complex distributions of frequency parameters is a more distant goal. Another interesting direction we are pursuing is the exploration of the consequences of disorder in the coupling parameter $a$. Finally, we note that a two-state version of this model (which of course does not lead to phase synchronization as discussed here) has recently been shown to accurately capture the unique statistics of blinking quantum dots [15]. Such wider applicability of the model, together with its analytic and numerical tractability, clearly opens the door to a number of new directions of investigation.

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